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| **CS 229r - Spectral Graph Theory in Computer Science** | Prof. Salil Vadhan |
| Problem Set 5 | |
| *Harvard SEAS - Fall 2020* | *Due: Fri Dec. 4, 2020 (5pm)* |

Your problem set solutions must be typed (in e.g. LaTeX) and submitted on Gradescope. You are allowed 12 late days for the semester, of which at most 5 can be used on any individual problem set. (1 late day = 24 hours exactly). Please name your file `ps5-lastname.*`.

The problem sets may require a lot of thought, so be sure to start them early. You are encouraged to discuss the course material and the homework problems with each other in small groups (2-3 people). Identify your collaborators on your submission. Discussion of homework problems may include brainstorming and verbally walking through possible solutions, but should not include one person telling the others how to solve the problem. In addition, each person must write up their solutions independently, and these write-ups should not be checked against each other or passed around.

Strive for clarity and conciseness in your solutions, emphasizing the main ideas over low-level details. Do not despair if you cannot solve all the problems! Difficult problems are included to stimulate your thinking and for your enjoyment, not to overwork you.

**Problem 1.(FOCS)** In a few sentences, describe what FOCS talks or activities you attended or watched, and what you learned from doing so.

**Problem 2.(Simpler proof of Leverage Score Theorem)** Let $G$ be an undirected, weighted, and connected graph. Spielman Theorem 13.5.1 states that for every edge $e = (a, b)$ in $G$, the leverage score $\ell_e = w_e \cdot R_{\text{eff}}(a, b)$ equals the probability that $e$ is in a random spanning tree $T$ of $G$, where $T$ is chosen with probability proportional to the product of its edge weights.

In Spielman and in class, we saw a rather complicated proof of this theorem. In this problem, you will see a simpler proof.

1. Show that
$$\Pr[e \notin T] = \frac{\sigma_{n-1}\left(L_G - w_e(\delta_a - \delta_b)(\delta_a - \delta_b)^T\right)}{\sigma_{n-1}\left(L_G\right)}.$$

2. Show that
$$\frac{\sigma_{n-1}\left(L_G - w_e(\delta_a - \delta_b)(\delta_a - \delta_b)^T\right)}{\sigma_{n-1}\left(L_G\right)} = 1 - \ell_e.$$

**Problem 3.(Gaussian elimination is slow on expanders)** We saw in class that Richardson iterations can solve Laplacian systems $Lx = b$ to within error $\varepsilon$ on expander graphs $G$ in just $O(\log(1/\varepsilon))$ applications of $L$, and hence in time $O(n \log(1/\varepsilon))$ for a constant-degree expander.

In contrast, here you will prove that Gaussian elimination takes time $\Omega(n^3)$ on any (regular, undirected) graph $G$ with spectral expansion at least $1 - \omega$ for a sufficiently small constant $\omega$. Suppose we eliminate vertices of $G$ in an arbitrary order $a_1, \ldots, a_n$. Recall that when we eliminate a vertex $a_i$, we replace it with a weighted clique on its neighbors. The time to perform Gaussian elimination is proportional to $\sum_{i=1}^{n} f(i)^2$, where $f(i)$ is the number of distinct neighbors of vertex

$a_i$ *at the time that it is eliminated*, which equals the number of distinct neighbors of vertex $a_i$ in the graph whose Laplacian is the Schur complement of $L$ with respect to $\{a_1, \ldots, a_{i-1}\}$.

1. Prove that there is a subset $C \subseteq \{a_1, \ldots, a_{n/2}\}$ of size at least $n/3$ such that the induced subgraph of $G$ on $C$ is connected. (Hint: show that if not, there are sets $S$ and $T$ of vertices, both of size between $n/6$ and $n/3$, with no edges between $S$ and $T$.)

2. Show that $\sum_i f(i)^2 = \Omega(n^3)$. (Hint: argue that $C$ has $\Omega(n)$ neighbors among $\{a_{n/2+1}, \ldots, a_n\}$.)

**Problem 4.(Spectral approximation of digraphs)** Recall that our notion of spectral approximation uses the Loewner order and thus only applies to symmetric matrices and undirected graphs. For an asymmetric matrix $A$, we can define its *undirectification* to be the matrix $U_A = (A + A^T)/2$. You previously encountered this matrix on Problem Set 1 (in the solution to Problem 3.1). Consider the following notion of approximation for possibly asymmetric $n \times n$ matrices $A$ and $B$ such that $U_A$ and $U_B$ are psd (which holds, for example, when $A$ and $B$ are Laplacians of Eulerian digraphs). We write $B \approx_\varepsilon A$ if for all vectors $x, y \in \mathbb{R}^n$, we have:

$$\left|x^T(A-B)y\right| \leq \varepsilon \cdot \|x\|_{U_A} \cdot \|y\|_{U_A}.$$

Throughout, assume that $A$, $B$, $U_A$, and $U_B$ are all invertible (which we can assume in Laplacian applications by restricting to $\vec{1}^\perp$ as usual).

1. Prove that $B \approx_\varepsilon A$ iff $\|U_A^{-1/2}(A-B)U_A^{-1/2}\| \leq \varepsilon$, which for symmetric matrices $A$, $B$ is equivalent to our usual notion of approximation:

$$(1-\varepsilon) \cdot A \preceq B \preceq (1+\varepsilon) \cdot A,$$

2. Show that

$$U_{A^{-1}}^{-1} = A^T U_A^{-1} A \succeq U_A.$$

   (Hint: write $A = U_A + V_A$ for a skew-symmetric matrix $V_A$.)

3. Show that for every $n \times n$ matrix $M$, we have

$$\|M\|_{U_A} \leq \left\|U_A^{-1/2} A M U_A^{-1/2}\right\|.$$

   (Hint: use the fact that the spectral norm of a matrix $N$ equals the square root of the largest eigenvalue of $N^T N$.)

4. Prove that if $B \approx_\varepsilon A$, then

$$\left\|I - A^{-1}B\right\|_{U_A} \leq \varepsilon.$$

5. Suppose that $A$ is a Laplacian of a connected Eulerian digraph $G$ (restricted to $\vec{1}^\perp$ to be made invertible) and we can construct a preconditioning matrix $B$ such that (a) $A \approx_{1/2} B$ (we've intentionally switched the roles of $A$ and $B$ from above, which can be done at a small increase in the approximation error) and (b) we can apply $B^{-1}$ to vectors in time $T$. Conclude that given a vector $b = Ax$ for $x \perp \vec{1}$, we can find a solution $\hat{x} \perp \vec{1}$ such that $\|\hat{x} - x\| \leq \varepsilon \|x\|$ (in standard Euclidean norm) in time $O(T \cdot \log(n w_{max}/w_{min}\varepsilon))$, where $w_{max}$ and $w_{min}$ are the maximum and minimum edge weights in $G$.