CS 229r - Spectral Graph Theory in Computer Science	Prof. Salil Vadhan

Problem Set 4

Harvard SEAS - Fall 2020 | *Due: Fri. Oct. 23, 2020 (5pm)*

Your problem set solutions must be typed (in e.g. LaTeX) and submitted on Gradescope. You are allowed 12 late days for the semester, of which at most 5 can be used on any individual problem set. (1 late day = 24 hours exactly). Please name your file `ps4-lastname.*`.

The problem sets may require a lot of thought, so be sure to start them early. You are encouraged to discuss the course material and the homework problems with each other in small groups (2-3 people). Identify your collaborators on your submission. Discussion of homework problems may include brainstorming and verbally walking through possible solutions, but should not include one person telling the others how to solve the problem. In addition, each person must write up their solutions independently, and these write-ups should not be checked against each other or passed around.

Strive for clarity and conciseness in your solutions, emphasizing the main ideas over low-level details. Do not despair if you cannot solve all the problems! Difficult problems are included to stimulate your thinking and for your enjoyment, not to overwork you.

Problem 1.(Final Project Proposal, due Sun. 11/8) Submit a couple of pages giving a detailed description of what your final project will look like. You should be able to clearly state your research questions, briefly articulate how your project relates to what has been done in the past, describe the approach you are taking, give your timeline for completing various aspects of the project, and discuss your fallback plan in case the project doesn't go as you hope.

Problem 2.(Limits on Spectral Expansion) Let G be a d-regular unweighted, undirected graph on n vertices and let T_d be the infinite d-regular tree with root r (so the root has degree d and every vertex has $d-1$ children and 1 parent). For a graph H, vertex a and $\ell \in \mathbb{N}$, let $p_\ell(H; a)$ denote the probability that if we do a random walk of length 2ℓ started at a, we end back at vertex a.

1. Show that for every vertex a of G, we have $p_\ell(G; a) \geq p_\ell(T_d; r) \geq C_\ell \cdot (d-1)^\ell / d^{2\ell}$, where C_ℓ is the ℓ'th Catalan number, which equals the number of properly parenthesized strings in $\{(,)\}^{2\ell}$ — strings where no prefix has more)'s than ('s.
2. Show that $\sum_a p_\ell(G; a) \leq 1 + (n-1) \cdot \omega(G)^{2\ell}$.
3. Using the fact that $C_\ell = \binom{2\ell}{\ell}/(\ell+1)$, prove that

$$\omega(G) \geq \frac{2\sqrt{d-1}}{d} - o(1),$$

   where the $o(1)$ term vanishes as $n \to \infty$ (and d is held constant).
4. Extra credit: figure out a generalization of this theorem and proof to directed graphs. (This is intentionally open-ended; we don't know what the best answer is and it could turn into an interesting research problem!)

Problem 3.(The Derandomized Square) In this problem, you'll see another deterministic (nearly) logpsace algorithm for Undirected S-T Connectivity, which is a closer parallel to the $O(\log^2 n)$ space repeated squaring algorithm we saw. and uses an operation that will come up again later in the course.

Let G be a d-regular digraph on n vertices, and H a c-regular digraph on d vertices. We construct the *derandomized square* of G with respect to H to be the following dc-regular digraph $\widetilde{G^2}$ on n vertices: For a vertex $u \in [n]$ and edge label $(i, j) \in [d] \times [c]$, we obtain the (i, j)-th neighbor of u in G through the steps:

I. Let v be the i-th neighbor of u in G, and let i' be such that the i-th edge leaving u is the i'-th edge entering v.

II. Let i'' be the j-th neighbor of i' in the H.

III. Let w be the i''-th neighbor of v in G.

1. Prove that if G has spectral expansion at least $\gamma = 1 - \omega$ and H has spectral expansion at least θ, then $\widetilde{G^2}$ has spectral expansion at least $\theta \cdot (1 - \omega^2)$. (Hint: write the random-walk matrix for $\tilde{G}^2$ in the form $P(I_n \otimes W_H)L$, where W_H is the random-walk matrix for H, and apply matrix decomposition to W_H.)

2. Suppose we define a sequence of graphs $G_0, G_1, \ldots$, where G_0 is obtained by adding self-loops to G to make it regular and aperiodic, and G_{t+1} is a derandomized square of G_t with respect to graphs H taken from an infinite family of c-regular expanders with spectral expansion at least θ. (Note that we use a different H at each step as the degrees of the G_t's are growing.) Show that for some constant $\theta < 1$ and some $t = O(\log n)$, it is ensured that every connected component of G_t has spectral expansion $\Omega(1)$.

3. Let $G_0, G_1, \ldots$ be as in Part 2. Using a sufficiently explicit family of constant-degree expanders H, it can be shown that neighbors in G_t can be computed in space $O(\log n + t) = O(\log n)$. Argue that S-T connectivity in G_t and hence G can be decided in space $O(\log n \cdot \log \log n)$. (Hint: use the space-efficient repeated squaring algorithm discussed in class.)

4. Extra credit: come up with a way to reduce the space complexity to $O(\log n)$ by carrying out a few more derandomized squares with graphs that are powers of graphs in our expander family to reduce the diameter to 1.

Problem 4.(Monotonicity of Current) Consider an (undirected, as always) resistor network $G = (V, E)$ with resistances $r : E \to \mathbb{R}^+$.

1. Suppose that we send one unit of external current from a to b. Let v be the vector of voltages we obtain, and let $\hat{v}$ be the vector of voltages if we remove the edge (a, b). (Assume that this does not disconnect the network.) Show that for every two vertices s and t,

$$v(s) - v(t) = \frac{\hat{v}(s) - \hat{v}(t)}{1 + R/r_{a,b}},$$

where R is the effective resistance between a and b when the edge (a, b) is removed. (Hint: view the network as composing in parallel the edge (a, b) with the rest of the network.)

2. Suppose we instead send one unit of external current from s to t. Show that if we decrease the resistance $r_{a,b}$ on edge (a, b), then the magnitude of current flowing on edge (a, b) cannot decrease. (Hint: express both the current here and the voltage differences in Part 1 in terms of L^+.)