CS 229r Spectral Graph Theory in Computer Science, Lecture 16 Oct 27, 2020

Lecture 16

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1 Recap

1.1 Resistor Networks

Recall the physical model of an undirected, weighted, and connected graph G in which we treat every edge as a resistor. If we view each edge e with weight w_e as an electrical conductance, the edge has resistance $r_e = \frac{1}{w_e}$. We can relate voltage, current, and resistance with Ohm's Law.

Definition 1 (Ohm's Law)**.** *Ohm's Law ($V = IR$) tells us that the voltage across a resistor (V) is equal to the current flowing over the resistor (I) times the resistance (R). Given a graph with voltages on the vertices and resistances on edge weights, the current that flows on each edge from a to b is* $\boldsymbol{i}(a,b) = \frac{v(a)-v(b)}{r_{(a,b)}} = w_{(a,b)}(v(a) - v(b))$*.*

Note that current is a signed value, $\boldsymbol{i}(b,a) = -\boldsymbol{i}(a,b)$, where a positive current corresponds to current flow from high voltage to low voltage.

Definition 2 (Net current)**.** i_{ext} *is the net current from vertex a is given by* $\sum_b w_{(a,b)}(v(a)-v(b)) = (Lv)(a)$*, which is the Laplacian applied to the voltage vector evaluated at vertex a.*

See the previous lecture notes (Lecture 16) for an example of a small resistor network.

Definition 3 (Flow conservation)**.** *Flow conservation (conservation of current) tells us that the net current at each a must equal zero if there are no external currents involved with the circuit. If there are nonzero entries on the net current $i_{ext} = Lv \neq 0$, there must be an external current that is fed in or taken out of the circuit.*

Definition 4 (Harmonic at vertex a)**.** *Since $(Lv)(a) = ((D - M)v)(a) = 0$, this implies that $(Dv)(a) = (Mv)(a)$. Simplifying, we have that $v(a) = \frac{1}{d(a)} \sum_{b\sim a} w_{a,b}v(b)$. We define this condition as v being harmonic at a.*

In other words, the voltage at a is the weighted average of the voltages at the neighbors of a. In mathematics, a harmonic functions are the solutions of Laplace's equations. They have the mean-value property that the average of a ball or sphere is equal to the value at its center. In our context, we have replaced spheres around a point with the neighborhood of a vertex.

We can also see the harmonic property by examining the stationary distribution of a random walk. We can say that at every node a, the random walk Laplacian at the stationary distribution is 0. We get

$$\begin{aligned} ((I - W)\pi)(a) &= 0 \\ \pi(a) &= (MD^{-1}\pi)(a) \\ \pi(a) &= \sum_{b\sim a} \frac{w_{b,a}\pi(b)}{d(b)} \end{aligned}$$

which is similar to the harmonic property at a in which the flow in equals the flow out. This is also the weighted average of neighbors. In the definition of harmonic, we are dividing by the degree of a outside the sum. Here, we are dividing by the degree of the neighbors inside the sum. The difference comes from the fact that the Laplacian and the random walk Laplacian differ by the D^{-1} matrix.

As we saw in the previous lecture, $i_{\text{ext}} = Lv \neq \mathbf{0}$ requires external currents in order to satisfy flow conservation. We denote $i_{\text{ext}}(a)$ to be the external current entering a. We note that $i_{\text{ext}} \perp \vec{1}$ which tells us that the current coming in has to equal the current leaving, so the global current is conserved.

In the internal nodes, the vector voltages are harmonic. On the boundary, the voltages are not harmonic.

1.2 Effective Resistance

Given any $i_{\text{ext}} \in \text{im}(L)$, where the image of L are all vectors orthogonal to $\vec{1}$, $v = L^{+} i_{\text{ext}}$ are induced voltages. The solution is not unique, but the kernel is adding constants. Thus, we only care about the potential difference, or differences in voltage.

We want to think of the entire resistor network as a single resistor. This leads us to the definition of the effective resistance.

Definition 5 (Effective resistance between a and b)**.**

$$\begin{aligned} R_{\mathit{eff}}(a,b) &= [v(a) - v(b) \text{ when } i_{ext} = \delta_a - \delta_b] \\ &= (L^{+} i_{ext})(a) - (L^{+} i_{ext})(b) \\ &= (\delta_a - \delta_b)^T L^{+} (\delta_a - \delta_b) \\ &= \|L^{+/2}\delta_a - L^{+/2}\delta_b\|^2. \end{aligned}$$

This gives us another motivation to calculate L^{+} efficiently (in linear time), as it gives us a quick way to read off effective resistances in any resistor network.

2 Breakout Exercise: Finding the Effective Resistance

Derive effective resistances between a and b in the following networks by finding voltages v inducing current $i_{ext} = \delta_a - \delta_b$ (i.e. $Lv = \delta_a - \delta_b$).

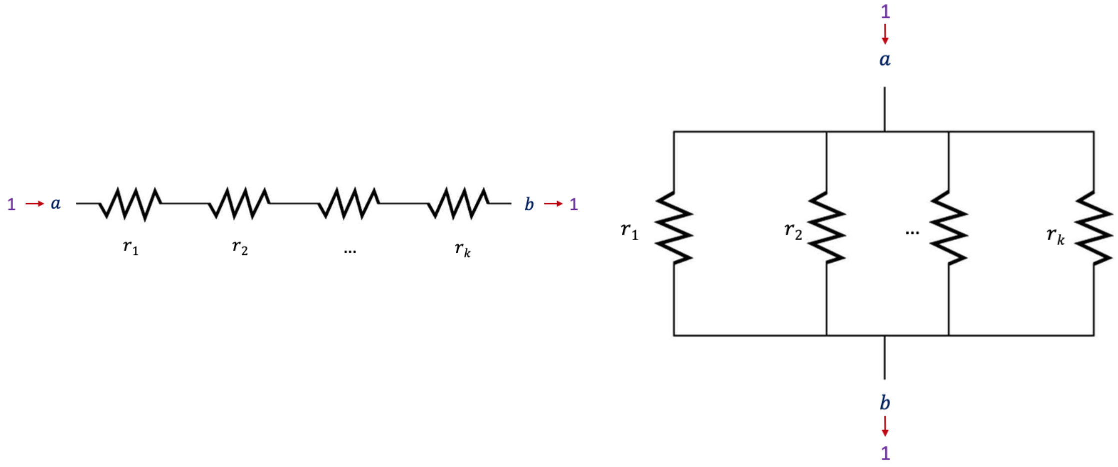

Figure 1: Series (left) and parallel (right) circuits.

For the series circuit, Ohm's Law tells us that if we start at node a with voltage 0, our next voltage is $V = IR = -1 * r_1 = -r_1$. Therefore, the voltages are $0, -r_1, -(r_1 + r_2), \ldots, -(r_1 + \cdots + r_k)$ at the nodes from a to b.

This assignment of voltages is harmonic at every node, since $-r_1 = \frac{1}{\frac{1}{r_1}+\frac{1}{r_2}} \cdot \left(\frac{0}{r_1} + \frac{-(r_1+r_2)}{r_2}\right)$. So for a series circuit, $R_{\text{eff}}(a,b) = r_1 + \cdots + r_k$.

For the parallel circuit, we have that the edge weights are $w_i = \frac{1}{r_i}$ for node i. We thus have edges $w_1 = \frac{1}{r_1}, ..., w_k = \frac{1}{r_k}$. When we have parallel edges, we sum edge weights $w_{(a,b)} = w_1 + \cdots + w_k$. Thus, the effective resistance is $\frac{1}{R_{\text{eff}}(a,b)} = \frac{1}{r_1} + \cdots + \frac{1}{r_k}$, or $R_{\text{eff}}(a,b) = \frac{1}{\frac{1}{r_1}+\cdots+\frac{1}{r_k}}$. This is the parallel law of circuits.

3 Schur Complements

Given $v(B)$ and constraint that v harmonic on $V - B$, find $i_{ext}(B)$. Suppose someone tells us what the voltages are on boundary, and the internal nodes are harmonic. We want to find the external current that will support these voltages. We can write down a formula for this matrix that will take us from voltages on B to currents on B using the Schur complement of the Laplacian, to eliminate the rows and columns corresponding the internal nodes.

Theorem 1: $i_{ext}(B) = L_B v(B)$ for $L_B = L(B,B) - L(B,S)L(S,S)^{-1}L(S,B)$. This is the Schur complement of L with respect to S or "on B" where $L = \begin{bmatrix} L(S,S) & L(S,B) \\ L(B,S) & L(B,B) \end{bmatrix}$.

Proof. We must first show that $L(S,S)^{-1}$ is invertible.

We know the Laplacian L is not invertible, so it doesn't have a kernel. Now, if we take the strict principal submatrix of Laplacian and check that it is strictly positive definite. We get that

$$x_S^T L(S,S) x_S = \begin{bmatrix} x_S \\ 0 \end{bmatrix}^T L \begin{bmatrix} x_S \\ 0 \end{bmatrix} > 0$$

because $\begin{bmatrix} x_S \\ 0 \end{bmatrix} \notin \text{Span}(\vec{1})$. $L(S,S)$ does not have a trivial kernel because otherwise $\begin{bmatrix} x_S \\ 0 \end{bmatrix}$ would be a vector of all zeros, which is in $\text{Span}(\vec{1})$.

Now, we want to show that $i_{ext}(B) = L_B v(B)$ for $L_B = L(B,B) - L(B,S)L(S,S)^{-1}L(S,B)$. Given $v(B)$, we define $v(s) = -L(S,S)^{-1}L(S,B)v(B)$. We now have $\begin{bmatrix} L(S,S) & L(S,B) \\ L(B,S) & L(B,B) \end{bmatrix} \begin{bmatrix} v(S) \\ v(B) \end{bmatrix} = \begin{bmatrix} 0 \\ L_B v(B) \end{bmatrix}$.

Now that we put this value of $V(S)$, we see that we get $L_B v(B) = -L(B,S)L(S,S)^{-1}L(S,B)v(B) + L(B,B)v(B)$. Since current is voltage over resistance ($I = V/R$) and L_B represents $1/R$, we have that $i_{ext}(B) = L_B v(B)$, which is what we wanted to prove.

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Theorem 2. L_B is a matrix obtained from L by using Gaussian elimination to eliminate S.

Proof. Consider S being a single element of node ($S = \{1\}$) and B being the rest of nodes ($B = \{2, \cdots, n\}$). By assuming node 1 being harmonic, write out the in matrix form equation as:

$$\begin{bmatrix} L(1,1) & L(1,B) \\ L(B,1) & L(B,B) \end{bmatrix} \begin{bmatrix} v(1) \\ v(B) \end{bmatrix} = \begin{bmatrix} 0 \\ i_{ext}(B) \end{bmatrix}.$$

We zero out everything in first column except $L(1,1)$ by row operation:

$$\begin{bmatrix} L(1,1) & L(1,B) \\ 0 & L(B,B) - \frac{L(B,1)L(1,B)}{L(1,1)} \end{bmatrix} \begin{bmatrix} v(1) \\ v(B) \end{bmatrix} = \begin{bmatrix} 0 \\ i_{ext}(B) \end{bmatrix}.$$

We can see that $L(B,B) - \frac{L(B,1)L(1,B)}{L(1,1)}$ in the right down corner is exactly the Schur Complements.

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Theorem 3. L_B is a Laplacian.

Proof. L_B and undirected Laplacian. The three conditions needs to be justify for being Laplacian of a weighted undirected graph. (1) The matrix is symmetric, (2) Off-diagonal elements are non-positive (3) The row and column sums are zero. We show these statements are true by inductively generalized from case where $|S| = 1$. Consider the original Laplacian:

$$\begin{bmatrix} d(1) & w_{12} & \dots & w_{1b} & \dots & w_{1n} \\ \vdots & & \ddots & & & \vdots \\ w_{b1} & w_{b2} & \dots & d(b) & \dots & w_{bn} \\ \vdots & & \vdots & & & \vdots \end{bmatrix}.$$

Again, we use row operation to zero out first column for every row except first row:

$$\begin{bmatrix} d(1) & w_{12} & \dots & w_{1b} & \dots & w_{1n} \\ \vdots & & \ddots & & & \vdots \\ 0 & w_{b2} - \frac{w_{1,2}w_{1,b}}{d(1)} & \dots & d(b) - \frac{w_{1,b}^2}{d(1)} & \dots & w_{bn} - \frac{w_{1,n}w_{1,b}}{d(1)} \\ \vdots & & \vdots & & & \vdots \end{bmatrix}.$$

For (3), since the sum of first row is zero, multiply it by any scale and add to the other row does not change the row sum. Thus other rows sum will remain zero after the above row operation. As for (1), we can check the symmetries of the above expression. Or alternatively check Schur Complements formula $L_B = L(B,B) - L(B,S)L(S,S)^{-1}L(S,B)$ is symmetric. And for (2), since every w is non-positive we can easily check the off diagonal elements of the above expression are non-positive as well.

It is also useful to think what happens in the corresponding graph. We replace every edge in vertex 1 forming the star with a clique and self loop of the rest of nodes as figure 2 shows.

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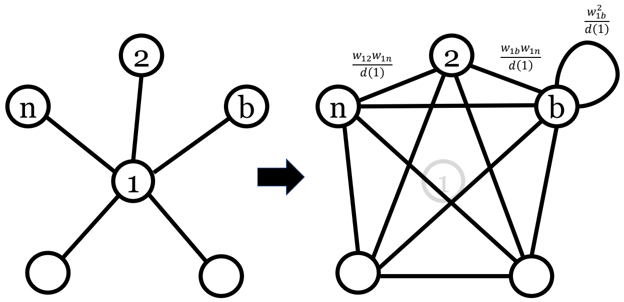

Figure 2: Equivalent network change for Schur complements.

We're adding a new edge with $\frac{w_{12}w_{1b}}{d(1)}$, etc.

This is related to the current problem set with derandomized squares where instead of taking clique, you are replacing that with an expander.

3.1 Random walk interpretation of Schur complements

Consider the random walk normalized Laplacian:

$$L_{rw} = I - W = \begin{bmatrix} I_S - W(S,S) & -W(S,B) \\ -W(B,S) & I_B - W(B,B) \end{bmatrix}.$$

The diagonal elements are principle sub-matrix of the random walk matrix in S and B. The off-diagonal elements are the random walk matrix flow between S and B.

$$\begin{aligned}\textit{Schur complement wrt } S &= L_{rw}(B,B) - L_{rw}(B,S)L_{rw}(S,S)^{-1}L_{rw}(S,B)\\ &= I_B - W_{BB} - (-W_{BS})(I_S - W_{SS})^{-1}(-W_{SB})\\ &= I_B - (W_{BB} + W_{BS}(I_S + W_{SS} + W_{SS}^2 + W_{SS}^3 + \cdots)W_{SB}).\end{aligned}$$

Observing the elements of second term:

- W_{BB} is random walk of 1 step staying in B.
- W_{SB} is go from B to S in 1 step.
- $(I_S + W_{SS} + \cdots)$ is walk any number of steps in S.
- W_{BS} is return to B.

We can skip time for any of the portions you are in S and treat passing through S as one timestep. This is a random walk matrix and its stationary distribution will be π conditioned on being in B.

Theorem 6. *R_{eff} is a metric, i.e.,*

1. *$R_{\mathit{eff}}(a,b) \geq 0$ for all a, b with equality iff $a = b$*
2. *$R_{\mathit{eff}}(a,b) = R_{\mathit{eff}}(b,a)$ for all a, b*
3. *$R_{\mathit{eff}}(a,b) \leq R_{\mathit{eff}}(a,b) + R_{\mathit{eff}}(b,c)$ for all a, b, c. This holds by the triangle inequality, since we know that square roots satisfy triangle inequality.*

Proof. The first and second terms are strait forward because the Laplacian matrix is positive semidefinite. Is is sufficient to look at the case in figure **??** because we can use Schur complements to eliminate all vertices except a, b, c. This preserves $R_{\text{eff}}(a,b)$, $R_{\text{eff}}(a,c)$, $R_{\text{eff}}(b,c)$.

Now, we only need to consider a network like the following:

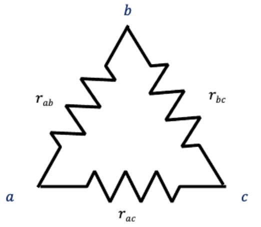

Figure 3: Triangle circuit to show the triangle inequality.

By series and parallel composition,

$$R_{\text{eff}}(a,b) = \frac{1}{\frac{1}{r_{a,b}} + \frac{1}{r_{bc}+r_{ac}}}$$

.

We can do the same for $R_{\text{eff}}(b,c)$ and $R_{\text{eff}}(a,c)$ and check if $R_{\text{eff}}(a,b) \leq R_{\text{eff}}(a,b) + R_{\text{eff}}(b,c)$ holds regardless the value of these three resistance.

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