CS 229r Spectral Graph Theory in Computer Science, Lecture 15	Oct 22, 2020
Lecture 15	
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1 Recap: Operations on Graphs

An (n, d, γ) graph has n vertices, d degree, and spectral expansion $\geq \gamma$. We are interested in constructing a family of constant-degree graphs with with good spectral expansion.

We will use the following graph operations to build expanders. Each operation improves one aspect of the expander at the cost of another one.

1. Squaring:
   - $(n, d, \gamma) \mapsto (n, d^2, 2\gamma - \gamma^2)$
   - $(n, d, 1-\omega) \mapsto (n, d^2, 1-\omega^2)$
2. Tensoring:
   - $(n, d, \gamma) \mapsto (n^2, d^2, \gamma)$,
   - $(n, d, 1-\omega) \mapsto (n^2, d^2, 1-\omega)$
3. Zig-zagging:
   - $(n, d_1, \gamma_1) ⓩ (d_1, d_2, \gamma_2) \mapsto (nd_1, d_2^2, \gamma_1\gamma_2^2)$
   - $(n, d_1, 1-\omega_1) ⓩ (d_1, d_2, 1-\omega_2) \mapsto (nd_1, d_2^2, 1-(\omega_1 + 2\omega_2))$

2 Explicit Construction of Expanders

(*Mildly Explicit Construction*) We start out with some d-regular graph with a large number of vertices. Specifically, we can use $H = (d^4, d, \frac{7}{8})$ and proceed to define

$$G_1 = H^2$$

$$G_{t+1} = G_t^2 ⓩ H.$$

Let n_t, d_t, and ω_t be the number of vertices, the degree, and the expansion of the graph G_t. By induction, we will have

$$n_{t+1} = d^4 \cdot n_t = d^{4(t+1)}$$

$$d_{t+1} = d^2$$

$$\omega_{t+1} \leq \omega_t^2 + 2(\frac{1}{8}) \leq \frac{1}{2}.$$

Let $\text{Time}(G_{t+1})$ be the time needed to compute neighbors of vertices in G_{t+1}. Then $\text{Time}(G_{t+1} = 2\text{Time}(G_t) + O(\log n_t)$, because it takes two evaluations of neighbors in G_t, two steps in H (which is constant), some constant time to compute steps in H, and some $\text{poly}(\log(n_{t+1}))$ time for string manipulation. For $\text{Time}(G_t)$ is polynomial in n_t. However, to show that these expanders are fully explicit, we are looking for log-poly time.

To get a tighter bound and achieve a fully explicit construction, we can add an additional tensoring operation into the graph generation to increase n_t.

3 Undirected S-T Connectivity in Logspace

Problem: given $G = (V, E)$ with $s, t \in V$, is there a path from s to t? Linear-time BFS/DFS solutions exist, but how can we reduce the space needed for this calculation?

3.1 Directed Graphs

For directed G, there exists an algorithm using space $O(\log^2(n))$, which uses repeated squaring to check entries of the M^n adjacency matrix. Specifically, $M^n(s,t) \neq 0 \implies$ there exists a path of at most n that takes us from s to t. This is effectively exploring all possible paths.

Observing

$$M^n(s,t) = \sum_{a \in V} M^{n/2}(s,a) M^{n/2}(a,t)$$

allows us to implement this recursively.

For each $s \in V$, we can recursively check if $M^{n/2}(s,a) \neq 0$ and if $M^{n/2}(a,t) \neq 0$. The depth of recursion will be $O(\log n)$, and all we need to keep track of on the stack is also $O(\log n)$ (info like current index in the loop). So space complexity is $O(\log^2 n)$, but the time complexity $2^{O(\log^2 n)} = O(n^{\log n})$ is quite bad.

3.2 Undirected Graphs

In HW2, we saw that randomized algorithms work well for this problem on undirected graphs. Here, we will show that $s - t$ connectivity can be solved in $O(\log n)$ space in deterministic space.

Reingold's algorithm (2004):

For vertices s, t in G undirected with N vertices:

1. Let H be a fixed $(D^4, D, \frac{3}{4})$ graph for some constant D.
2. We reduce (G, s, t) to G_0, s_0, t_0, where G_0 is a d^2-regular and aperiodic modification of G, and s_0, t_0 are connected in G_0 if and only if s, t are connected in G.
3. We let $G_{t+1} = G_t^2 ⓩ H$, where H is a $(d^4, d, \frac{3}{4})$-graph. $s_0, t_0 \in G_0$ will be connected iff s, t connected in G.
4. The claim is that after $l = O(\log N)$ iterations, G_l will be a good enough expander that if we try all paths of length $O(\log N)$ from s_l, we will be able to reach t_l through one of them if s and t in G are connected. We are essentially using our operations to turn some arbitrary graph (that could be large) into an expander.

   The squaring roughly doubles γ, and even with the zig-zagging, which decreases expansion, we get

   $$\gamma_{t+1} \geq (2\gamma_t - \gamma_2^2) \cdot (\frac{3}{4})^2 \approx \frac{18}{16} \cdot \gamma_t$$

   for small γ, so γ for the component containing s_t (and in fact each component of the graph) is increasing with each step. In undirected graphs, the initial expansion will be at least $\frac{1}{\text{poly}(N)}$, so only logarithmically many constant multiplicative factor improvements of γ are needed for G_t to become a constant degree expander. Since constant degree expanders have logarithmic diameter, checking paths of length $O(\log N)$ will suffice for determining $s - t$ connectivity. The zig-zagging helps keep degree down and reduce space complexity.

Open Problem: Can randomized algorithms can be turned into deterministic ones with only a constant increase in space?

4 Resistor Networks

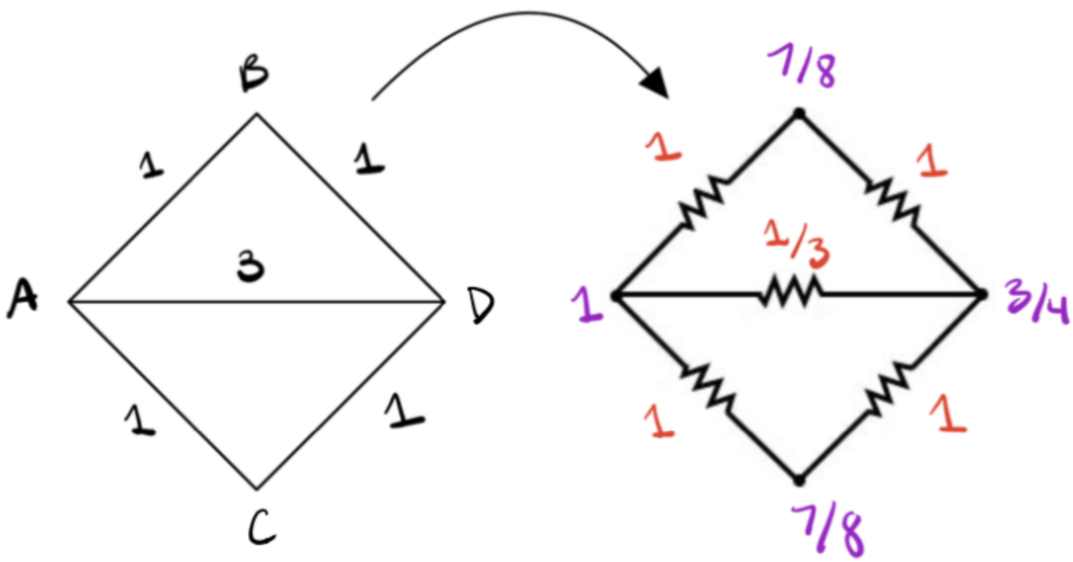
Consider G to be an undirected, weighted and connected graph where each edge e is a resistor with resistance $r_e = \frac{1}{w_e}$. Let $\mathbf{i}(a,b) = -\mathbf{i}(b,a)$ be the current flowing from a to b. Given voltages $\mathbf{v} : \mathbf{V} \to \mathbf{R}$ at the vertices, it follows from Ohm's Law ($V = IR$) that the current on edge (a,b) is

$$\mathbf{i}_{(a,b)} = \frac{v(a) - v(b)}{r_{(a,b)}} = w_{a,b} \cdot (v(a) - v(b)).$$

The net current from a is

$$\mathbf{i}_{ext}(a) = \sum_b w_{a,b}\dot{(}v(a) - v(b)) = (L\mathbf{v})(\mathbf{a}).$$

Example: (red = resistance, violet = voltage)



$$L = \begin{bmatrix} 5 & -1 & -1 & 3 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -3 & -1 & -1 & 5 \end{bmatrix}$$

$$\mathbf{i_{ext}} = \begin{bmatrix} 5 & -1 & -1 & 3 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -3 & -1 & -1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 7/8 \\ 7/8 \\ 3/4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

By flow conservation, net current at each vertex must equal zero. $L\mathbf{v} \neq \mathbf{0}$ means there must be external currents being applied to the system. $\mathbf{i_{ext}}(\mathbf{a})$ is the amount of external current entering a. The only nontrivial solutions to $L\mathbf{v} = \mathbf{0}$ are constant vectors. The external current vector will be orthogonal to the constant vector for current flowing into the network to equal the current flowing out.

For any $\mathbf{i_{ext}} \in \text{Im}(\mathbf{L})$ (these are vectors orthogonal to $\vec{1}$), the induced voltages will be $\mathbf{v} = \mathbf{L}^{+}\mathbf{i_{ext}}$. Since L is singular, solutions for the system are not unique.

5 Effective Resistance

The effective resistance between a and b is the resistance we get if we consider the entire network to be a single resistor between a and b.

We use Ohm's Law and consider adding one unit of current to a and removing one unit of current out from b, then measuring the potential difference:

$$\begin{aligned} R_{\text{eff}}(a,b) &= [v(a) - v(b) \text{ when } i_{ext} = \delta_a - \delta_b] \\ &= (L^+_{i_{ext}})(a) - (L^+_{i_{ext}})(b) \\ &= (\delta_a - \delta_b)^T L^+ (\delta_a - \delta_b) \\ &= ||L^{+/2}\delta_a - L^{+/2}\delta_b||^2 \end{aligned}$$

($R_{\text{eff}}(A, D)$ in the example diagram from earlier would be $\frac{1}{4}$.)

This offers another motivation for wanting to calculate L^+ efficiently.

Theorem: Finding effective resistance is related to optimizing $x^T Lx$ with additional constraints on x to avoid getting a trivial solution of 0.

$$\frac{1}{R_{\text{eff}}(a,b)} = \min_{x \in \mathbb{R}^v; x(a)=1; x(b)=0} (x^T Lx).$$

Proof sketch: Let $i_{ext} = \delta_a - \delta_b$ and $v = L^+ i_{ext}$. Then $x = \frac{v - v(b)\vec{1}}{R_{\text{eff}}(a,b)}$ because $x(b) = 0$ and $x(a) = \frac{v(a)-v(b)}{R_{\text{eff}}(a,b)} = 1$. We claim x is the unique minimizer. To prove this, we check that the partial derivatives of $x^T Lx$ are 0 at x, except for the a and bth coordinates, which are fixed. Then, we argue that $x^T Lx$ is strictly convex.

Rayleigh's Monotonicity Principle: If we decrease resistances (by increasing edge weights), effective resistances cannot increase.