CS 229r Spectral Graph Theory in Computer Science, Lecture 13 Oct 15, 2020

Lecture 13

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1 Agenda

- Expansion of random graphs
- Random walks on expanders
- Operations on exxpanders

2 Recap

Theorem: Let M be an infinite family of regular, undirected, lazy, constant degree graphs. The folloiwng are equivelent.

1. $\exists\gamma > 0$ s.t. every $G \in M$ has spectral expansion at least γ(ie.$\omega(G) = min\{\omega_2, -\omega_n\} \leq 1-\gamma$)
2. $\exists\epsilon > 0$ s.t. every $G \in M$ is a (n/2, ϵ) edge expansion(ie. $\forall S, |S| \leq n/2$, $|e(S, S^c)| \geq \epsilon * d * |S|$)
3. $\exists\epsilon > 0$ s.t. every $G \in M$ is a (n/2, $1+\delta$) vertex expansion(ie. $\forall S, |S| \leq n/2$, $|N(S)| \geq (1+\delta) * |S|$)

Expander Mixing Lemma
Let G be a regular digraph with spectral expansion $\gamma = 1-\omega$.We want to discuss such case that ω is small then γ is getting closer to 1. Then for every S,T$\subseteq V$with $|S| = \alpha n$ and $|T| = \beta n$,

$$|\frac{|e(S,T|}{E} - \alpha\beta| \leq \omega * \sqrt{\alpha(1-\alpha)\beta(1-\beta)}$$

3 Existence of Expanders

A uniformly random d-regular graph is a very good expander.
eg.
Vertex Expansion: ($\alpha_{d,\epsilon} * n$, d-1-ϵ) vertex expander, $\alpha_{d,\epsilon} > 0$, (n/2, 1+δ_α) - vertex expander, $\delta_d > 0$, $\delta_d -> 1$ as $d -> \infty$

Proof idea:

- For each set S of size k, argue Pr[does not expand enough $<< 1/\binom{n}{k}$]
- union bound over sets S

Spectral Expansion $\omega = \frac{2\sqrt{d-1}}{d} + \epsilon$ [Conjecture by Alon '86](note: $\frac{2\sqrt{d-1}}{d}$ is like the largest e-value of ∞ d-regular tree
Proof idea: $Tr(W^{2t}) = \sum_{j=1}^{n} \omega_j^{2t} \geq 1 + \omega(G)^2 t$
$Pr_G[\omega(G) \geq \omega] \leq \frac{E_G[Tr(W^{2t}]-1}{\omega^{2t}}$
$Tr(W^{2t}) = \sum_{a=1}^{n} W_{a,a}^{2t} = Pr_{randomwalk}$[walks starts and ends at a]
Then show, $Pr_{G,r.w}$[r.w. of length 2t, starts and ends at a]$\leq 1/n + 1/n^{1+c}$ for t=O(log n). Check Spielman Ch.8 for random dense graphs.
Ramanujan Graphs: $\omega \leq \frac{2\sqrt{d-1}}{d}$, no ϵ!

1. Not known that random graphs have this whp.
2. Explicit construction from deep number theory(replying on "Ramanujan Conjectures")
3. Bipartite Ramanujan graphs recently proved(after 2015) to exist using probabilistic argument that only establishes Pr$\geq$ 0.[See Spielman Part VII]

4 Random Walks on Expanders

Motivating example:
Power Method: M p.s.d with largest e-value μ_1

1. Choose random vector x $\{\pm 1\}^n$
2. output $y = M^k x$ for k=O($\frac{log(n/q}{q}$) w.p greaster than constant probablity 3/16(proved before)

$$\frac{y^T M y}{y^T y} \geq (1-\epsilon) * \mu_1$$

Reducing Failure probability

1. Repeat t times such that $x^{(1)}, ..., x^t \leftarrow \{\pm 1\}^n$
2. Compute y = Mx for each of x.
3. Output choose the y that maximize Rayleigh quotient: $y^T M y/(y^T y)$

Then we can have Pr$[y^T M y/(y^T y) < (1-\epsilon)\mu_1] \leq (13/16)^t = 2^{-\omega(n)}$ for $t = O(n)$
But, can we do it better(tossing coin is expensive)? *notes from Salil: any randomized can be expressed as deterministic algorithm(extreme version)*

- Choose $x^1, ..., x^t$ using a random walk on an expander G=$< V, E >$, $V = 2^n$, $V \leftrightarrow \{\pm 1\}^n$
- choose $x^{(1)}$ randomly chosen from V, $x^{(2)}$ randomly chosen from $x^{(1)}$'s d neighbors,...,$x^{(t)}$ randomly chosen from $x^{(t-1)}$'s d neighbors
  random bits = n+O(t*logd) = O(n), t = O(n), d=O(1)

B = $x \in \{\pm 1\}:$ for $y = M^k x, y^T M y/(y^T y) < (1-\epsilon)\mu_1$
$\mu = \mu(B) = \frac{|B|}{|V|}$
Theorem: if. G has spectral expasion $\gamma = 1 - \omega$, $V_1, V2, ..., V_t$ are random walk on G with uniform start vertex V_1 then

$$\begin{aligned} \forall B, Pr[\wedge_{j=1}^{t}(V_j \in B)] &\leq (\mu + \omega(1-\mu)^T \\ &\leq 2^{-\Omega(t)} \text{ for constants } \mu, \omega < 1 \end{aligned}$$

$\wedge_{j=1}^{t}(V_j \in B)$ means $V_1 \in B, V_2 \in B ... V_t \in B$
Proof: W = random walk matrix, P = diag($\overrightarrow{1}_B$), $\mu = \overrightarrow{1}/N$

$$\begin{aligned} Pr[\wedge_{j=1}^{t}(V_j \in B)] &= |(PW)^{t-1} P\mu|_1 \\ &= |(PWP)^{t-1} Pu|_1 \text{ because } P^2 = P \\ &\leq \sqrt{N} * ||(PWP)^{t-1} Pu|| \\ &\leq \sqrt{N} * ||PWP||^{t-1} ||P\mu|| \text{ because Cauchy-Schwartz inequality} \\ &\leq \sqrt{N} * (\mu + \omega(1-\mu))^t * \sqrt{\frac{\mu}{N}} \end{aligned}$$

Definition(spectral norm): $||M|| = max_{x\neq 0}\frac{||Mx||}{||x||}$ = largest singular value of M Matrix Decomposition:
Lemma: G has spectral expansion γ iff

$$W = \gamma J + (1-\gamma)E$$

where J = all 1/n matrix and $||E|| \leq 1$
Thus:

$$\begin{aligned} ||PWP|| &= ||\gamma PJP + (1-\gamma)PEP|| \\ &\leq \gamma||PJP|| + (1-\gamma)PEP|| \\ &\leq \mu + (1-\gamma) * \mathbf{1} \\ &= \mu + \omega(1-\mu) \end{aligned}$$

where PJPx = $(\sum_{j\in B} x_j) * \frac{\vec{1}}{N}$, P = diag($\overrightarrow{1_B}$)