# Lecture 12

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# 1 Agenda

Our main topic for the class is continuing with expander graphs. Specifically we will cover, (last two points were differed to the next lecture due to technical difficulties)

- Vertex expansion and the relation to spectral expansion (that was introduced last time).
- Expander mixing lemma (a useful property of expander graphs).
- Expansion of random graphs (deeper discussion than the reading).
- Random walks on expanders.

# 2 Recap: Expander Graphs

## 2.1 Spectral Expansion

An infinite family of $d$-regular graphs with degree $d$ that can be a function of $n$ but we will usually strive for $d = O(1)$, where $n \to \infty$. While our discussion will sometimes concern digraphs, it will more often revolve around undirected graphs and graphs will be unweighted throughout. The two main desirable properties is **sparse** ($d = O(1)$ or $d = \text{poly}(\log(n))$) and **well-connected** ($\gamma(G) = 1 - \omega(G) \geq \gamma$ for constant $\gamma > 0$) graphs[1]. In this case, with each step of the random walk we get closer by a constant factor to the stationary distribution $\mathbf{1}/n$ as the graph is regular.

## 2.2 $(k, a)$ Vertex Expansion

For all sets of vertices $S$ up to size $k$ (we think of $k$ as linear in $n$), the set of neighbors $N(S)$, is at least of size $a|S|$. Here, $a = 1 + \Omega(1)$, that is, a constant bigger than 1 independent of $n$. Intuitively, this captures the notion of expanding the number of vertices we can be on after each step of the random walk.

## 2.3 $(k, a)$ Edge Expansion

For all sets of vertices $S$ up to size $k$ we want $|e(S, S^c)| > \varepsilon d|S|$. In other words, we want the number of edges going through a cut to be large. This is equivalent to the conductance of sets of vertices up to some size $k$ (again, think of $k$ linear in $n$). We want the fraction of edges leaving to be at least a constant $\varepsilon = \Omega(1)$.

# 3 Spectral Expansion vs. Vertex Expansion

In the previous lecture we stated a relationship between spectral expansion and edge expansion for undirected graphs. In this lecture, we talk about a similar result for vertex expansion and digraphs:

**Theorem 1.** *Let $\mathcal{M}$ be a family of constant-degree regular* **digraphs**. *The following statements are equivalent.*

[1] Although in the class we used $\omega_\pi(G) = \max\{\omega_2, -\omega_n\}$ the standard notation used in the literature is $\omega_*(G)$.

*1. $\exists \gamma > 0$, such that every $G \in \mathcal{M}$ has spectral expansion at least $\gamma$.*

*2. $\exists \epsilon > 0$, such that every $G \in \mathcal{M}$ is an $(N/2, 1+\epsilon)$* **vertex** *expander.*

*Where* $(1) \Rightarrow (2)$ *even for unbounded degree and* $(2) \Rightarrow (1)$ *when* $d = O(1)$.

*Proof.*
$(2) \Rightarrow (1)$ in section.
$(1) \Rightarrow (2)$ Let $S \subseteq V$, $|S| = \alpha N$, $\alpha = 1/2$. The main idea is to start a random walk from uniform distribution on $S$ and look at what it means to get "$\gamma$ closer" to the stationary distribution $u = (\mathbf{1}/n)$ in terms of support. To that end, define $u_S = \frac{1}{|S|}\mathbb{1}_S$ to be the uniform on the set $S$, then we have that,

$$|N(S)| = |\text{supp}(Wu_S)| \geq \frac{1}{\text{CP}(Wu_S)}.$$

Where we define the *collision probability* of a probability vector $p$ as the probability of sampling the same point within two iid draws ($x$ and $x'$ below) from $p$. To see why the inequality holds, note that for any $p$ supported on $S$, $u_S \perp p$ and thus,

$$\begin{aligned}\text{CP}(p) = \Pr_{x,x'\sim p}[x = x'] = \sum_a p_a^2 = \|p\|^2 = \\ = \|u_S\|^2 + \|p - u_S\|^2 = \frac{1}{|S|} + \|p - u_S\|^2.\end{aligned}$$

Therefore for any $p$, $\text{CP(p)} = \frac{1}{n} + \|p - u\|^2$ and thus,

$$\begin{aligned}\text{CP}(Wu_S) - \frac{1}{n} &= \|Wu_S - u\|^2 \\ &\leq \omega(G)^2\|Wu_S - u\|^2 = \omega(G)^2\left(\text{CP}(u_S) - \frac{1}{n}\right)\end{aligned}$$

And so,

$$\frac{1}{|N(S)|} - \frac{1}{n} \leq (1-\gamma)^2\left(\frac{1}{|S|} - \frac{1}{n}\right),$$

here in the first inequality we use that $Wu = u$ and the definition of $\omega(G)$.

□

# 4 Breakout Question

Let $M$ adjacency matrix of a constant-degree expander on $n$ vertices.
Q: Which expansion properties are retained by the following graphs? (see Table 1).

- $M^2$, the graph corresponding to doing two steps of the random walk.
- $M + \log(n)I$, the graph corresponding to adding a *log* factor of self loops.
- Two disjoint copies of the same graph. Note that here (and in the following item) there is $n/3$ vertex expansion but not $n/2$ vertex expansion due to taking one copy of $M$ that is disjoint from the other.
- A bipartite graph of two copies of $M$ where in each step we move from one side to another.
- The graph corresponding to tensoring $M$ with the all 1 matrix.

**Table 1**: Preservation of expansion properties under different matrix operations.

| Adjacency Matrix | Spectral Expansion $\Omega(1)$ | Edge Expansion $(\frac{n}{2}, \Omega(1))$ | Vertex Expansion $(\frac{n}{2}, 1+\Omega(1))$ | Edge Expansion $(\frac{n}{3}, \Omega(1))$ | Vertex Expansion $(\frac{n}{3}, 1+\Omega(1))$ |
|---|---|---|---|---|---|
| $M^2$ | ✓ | $\longrightarrow$ | $\longrightarrow$ | $\longrightarrow$ | $\longrightarrow$ |
| $M + \log(n)I$ | $\omega \geq 1 - \frac{d}{\log n}$ ✗ | ✗ | ✓ | ✗ | ✓ |
| $\begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}$ | ✗ | ✗ | ✗ | ✓ | ✓ |
| $\begin{pmatrix} 0 & M \\ M & 0 \end{pmatrix}$ | ✗ | ✓ | ✗ | ✓ | ✓ |
| $\begin{pmatrix} M & M \\ M & M \end{pmatrix}$ | ✓ | ✓ | ✓ | ✓ | ✓ |

## 5 Expander Mixing Lemma

This discussion will be be reminiscent of conductance but we will not be only talking about the edges between a set and its complement but about edges between any two sets of vertices (we will denote that by $e(S, T)$).

**Lemma 1.** *Let $G$ be a regular digraph with spectral expansion $\gamma = 1 - \omega$. Then, for every $S, T \subseteq V$ with $|S| = \alpha n$ and $|T| = \beta n$,*

$$\left| \frac{|e(S,T)|}{|E|} - \alpha\beta \right| \leq \omega \cdot \sqrt{\alpha(1-\alpha)\beta(1-\beta)}. \tag{1}$$

*Proof.*

$$|e(S,T)| = u_T^\top M u_S \qquad M = \text{ adj. matrix}$$

$$\frac{|e(S,T)|}{|E|} = \frac{u_T^\top W u_S}{n} \qquad W = \text{ random walk matrix.} \tag{2}$$

We now decompose $u_T$ and $u_S$ to components along $u$ and perpendicular to $u$ (as is done in the spectral expansion implies vertex expansion theorem), $u_T = \beta n \vec{u} + u_T^\perp$ and $u_S = \alpha n \vec{u} + u_S^\perp$ and expand (2) into 4 terms. We have then,

$$(u_T^\perp)^\top W \vec{u} = \vec{u}^\top W u_S^\perp = 0$$
$$\beta n \vec{u}^\top W \alpha n \vec{u} = n\alpha\beta$$

$$\begin{aligned} |(u_T^\perp)^\top W u_S^\perp| &\leq \|u_T^\perp\| \cdot \|W u_S^\perp\| \\ &\leq \omega \cdot \|u_T^\perp\| \cdot \|u_S^\perp\|. \end{aligned}$$

where the first inequality is due to Cauchy-Schwarz and the second is due to expansion. Thus,

$$\left| \frac{|e(S,T)|}{|E|} - \alpha\beta \right| = \left| \frac{u_T^\top W u_S}{n} - \alpha\beta \right| \leq \frac{1}{n}\omega \|u_T^\perp\| \|u_S^\perp\|.$$

□

We conclude by noting that $u_S^\perp(a) = \begin{cases} \alpha & a \in S \\ 0 & \text{otherwise} \end{cases}$ and so $\|u_S^\perp\| = \sqrt{n\alpha(1-\alpha)}$. When we were talking about edge expansion, $T$ was $S^c$ and we were only discussing one direction of the inequality.

**Remark** (Converse Direction.) If we have inequality (1) for all sets $S, T$, then, $\omega(G) = O(\omega \log(1/\omega))$.