Lecture 12

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1 Agenda

Our main topic for the class is continuing with expander graphs. Specifically we will cover, (last two points were differed to the next lecture due to technical difficulties)

- Vertex expansion and the relation to spectral expansion (that was introduced last time).
- Expander mixing lemma (a useful property of expander graphs).
- Expansion of random graphs (deeper discussion than the reading).
- Random walks on expanders.

2 Recap: Expander Graphs

2.1 Spectral Expansion

An infinite family of d-regular graphs with degree d that can be a function of n but we will usually strive for $d = O(1)$, where $n \to \infty$. While our discussion will sometimes concern digraphs, it will more often revolve around undirected graphs and graphs will be unweighted throughout. The two main desirable properties is **sparse** ($d = O(1)$ or $d = \text{poly}(\log(n))$) and **well-connected** ($\gamma(G) = 1 - \omega(G) \geq \gamma$ for constant $\gamma > 0$) graphs[1]. In this case, with each step of the random walk we get closer by a constant factor to the stationary distribution $\mathbf{1}/n$ as the graph is regular.

2.2 (k, a) Vertex Expansion

For all sets of vertices S up to size k (we think of k as linear in n), the set of neighbors $N(S)$, is at least of size $a|S|$. Here, $a = 1 + \Omega(1)$, that is, a constant bigger than 1 independent of n. Intuitively, this captures the notion of expanding the number of vertices we can be on after each step of the random walk.

2.3 (k, a) Edge Expansion

For all sets of vertices S up to size k we want $|e(S, S^c)| > \varepsilon d|S|$. In other words, we want the number of edges going through a cut to be large. This is equivalent to the conductance of sets of vertices up to some size k (again, think of k linear in n). We want the fraction of edges leaving to be at least a constant $\varepsilon = \Omega(1)$.

3 Spectral Expansion vs. Vertex Expansion

In the previous lecture we stated a relationship between spectral expansion and edge expansion for undirected graphs. In this lecture, we talk about a similar result for vertex expansion and digraphs:

Theorem 1. *Let $\mathcal{M}$ be a family of constant-degree regular* **digraphs**. *The following statements are equivalent.*

[1] Although in the class we used $\omega_\pi(G) = \max\{\omega_2, -\omega_n\}$ the standard notation used in the literature is $\omega_*(G)$.

1. $\exists \gamma > 0$, *such that every* $G \in \mathcal{M}$ *has spectral expansion at least* γ.

2. $\exists \epsilon > 0$, *such that every* $G \in \mathcal{M}$ *is an* $(N/2, 1+\epsilon)$ **vertex** *expander.*

Where $(1) \Rightarrow (2)$ *even for unbounded degree and* $(2) \Rightarrow (1)$ *when* $d = O(1)$.

Proof.
$(2) \Rightarrow (1)$ in section.
$(1) \Rightarrow (2)$ Let $S \subseteq V$, $|S| = \alpha N$, $\alpha = 1/2$. The main idea is to start a random walk from uniform distribution on S and look at what it means to get "γ closer" to the stationary distribution $u = (\mathbf{1}/n)$ in terms of support. To that end, define $u_S = \frac{1}{|S|}\mathbb{1}_S$ to be the uniform on the set S, then we have that,

$$|N(S)| = |\text{supp}(Wu_S)| \geq \frac{1}{\text{CP}(Wu_S)}.$$

Where we define the *collision probability* of a probability vector p as the probability of sampling the same point within two iid draws (x and x' below) from p. To see why the inequality holds, note that for any p supported on S, $u_S \perp p$ and thus,

$$\begin{aligned}\text{CP}(p) &= \Pr_{x,x'\sim p}[x = x'] = \sum_a p_a^2 = \|p\|^2 = \\ &= \|u_S\|^2 + \|p - u_S\|^2 = \frac{1}{|S|} + \|p - u_S\|^2.\end{aligned}$$

Therefore for any p, CP(p) $= \frac{1}{n} + \|p - u\|^2$ and thus,

$$\begin{aligned}\text{CP}(Wu_S) - \frac{1}{n} &= \|Wu_S - u\|^2 \\ &\leq \omega(G)^2\|Wu_S - u\|^2 = \omega(G)^2\left(\text{CP}(u_S) - \frac{1}{n}\right)\end{aligned}$$

And so,

$$\frac{1}{|N(S)|} - \frac{1}{n} \leq (1-\gamma)^2\left(\frac{1}{|S|} - \frac{1}{n}\right),$$

here in the first inequality we use that $Wu = u$ and the definition of $\omega(G)$.

□

4 Breakout Question

Let M adjacency matrix of a constant-degree expander on n vertices.
Q: Which expansion properties are retained by the following graphs? (see Table 1).

- M^2, the graph corresponding to doing two steps of the random walk.
- $M + \log(n)I$, the graph corresponding to adding a *log* factor of self loops.
- Two disjoint copies of the same graph. Note that here (and in the following item) there is $n/3$ vertex expansion but not $n/2$ vertex expansion due to taking one copy of M that is disjoint from the other.
- A bipartite graph of two copies of M where in each step we move from one side to another.
- The graph corresponding to tensoring M with the all 1 matrix.

Table 1: Preservation of expansion properties under different matrix operations.

Adjacency Matrix	Spectral Expansion $\Omega(1)$	Edge Expansion $(\frac{n}{2}, \Omega(1))$	Vertex Expansion $(\frac{n}{2}, 1+\Omega(1))$	Edge Expansion $(\frac{n}{3}, \Omega(1))$	Vertex Expansion $(\frac{n}{3}, 1+\Omega(1))$
M^2	✓	$\longrightarrow$			
$M + \log(n)I$	$\omega \geq 1 - \frac{d}{\log n}$ ✗	✗	✓	✗	✓
$\begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}$	✗	✗	✗	✓	✓
$\begin{pmatrix} 0 & M \\ M & 0 \end{pmatrix}$	✗	✓	✗	✓	✓
$\begin{pmatrix} M & M \\ M & M \end{pmatrix}$	✓	✓	✓	✓	✓

5 Expander Mixing Lemma

This discussion will be be reminiscent of conductance but we will not be only talking about the edges between a set and its complement but about edges between any two sets of vertices (we will denote that by $e(S,T)$).

Lemma 1. *Let G be a regular digraph with spectral expansion $\gamma = 1 - \omega$. Then, for every $S, T \subseteq V$ with $|S| = \alpha n$ and $|T| = \beta n$,*

$$\left| \frac{|e(S,T)|}{|E|} - \alpha\beta \right| \leq \omega \cdot \sqrt{\alpha(1-\alpha)\beta(1-\beta)}. \tag{1}$$

Proof.

$$|e(S,T)| = u_T^\top M u_S \qquad M = \text{ adj. matrix}$$

$$\frac{|e(S,T)|}{|E|} = \frac{u_T^\top W u_S}{n} \qquad W = \text{ random walk matrix.} \tag{2}$$

We now decompose u_T and u_S to components along u and perpendicular to u (as is done in the spectral expansion implies vertex expansion theorem), $u_T = \beta n \vec{u} + u_T^\perp$ and $u_S = \alpha n \vec{u} + u_S^\perp$ and expand (2) into 4 terms. We have then,

$$(u_T^\perp)^\top W \vec{u} = \vec{u}^\top W u_S^\perp = 0$$
$$\beta n \vec{u}^\top W \alpha n \vec{u} = n\alpha\beta$$

$$\begin{aligned} |(u_T^\perp)^\top W u_S^\perp| &\leq \|u_T^\perp\| \cdot \|W u_S^\perp\| \\ &\leq \omega \cdot \|u_T^\perp\| \cdot \|u_S^\perp\|. \end{aligned}$$

where the first inequality is due to Cauchy-Schwarz and the second is due to expansion. Thus,

$$\left| \frac{|e(S,T)|}{|E|} - \alpha\beta \right| = \left| \frac{u_T^\top W u_S}{n} - \alpha\beta \right| \leq \frac{1}{n} \omega \|u_T^\perp\| \|u_S^\perp\|.$$

□

We conclude by noting that $u_S^\perp(a) = \begin{cases} \alpha & a \in S \\ 0 & \text{otherwise} \end{cases}$ and so $\|u_S^\perp\| = \sqrt{n\alpha(1-\alpha)}$. When we were talking about edge expansion, T was S^c and we were only discussing one direction of the inequality.

Remark (Converse Direction.) If we have inequality (1) for all sets S, T, then, $\omega(G) = O(\omega \log(1/\omega))$.