Announcements

- start recording
- scribe: work w/other scribe to produce one set of notes
- My OH: Mon 12:30–1:30, Thu 9–10
- TF hybrid section/OH: Mon 2–3, Wed 2–3, Wed 5–6, Thu 5:30–6:30
- PS 1 due Fri : note collaboration + late day policies
- If Zoom goes down, check Piazza
- Sync whiteboard
- gather after class : link in chat
- jamboard link in chat

Agenda

- Markov Chain Monte Carlo (cont.)
- Graph coloring
- Isoperimetry + Conductance + "Edge Expansion"

Correction from last time: If $1 = \mu_1 > |\mu_j|$ for $j = 2, \dots, n$

then by Jordan normal form, $W = SUS^{-1}$

for $U = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ & \mu_2 & & * \\ & & \ddots & \\ 0 & & & \mu_n \end{pmatrix}$ and $U^t \to \begin{pmatrix} 1 & & & 0 \\ & 0 & & \\ & & \ddots & \\ 0 & & & 0 \end{pmatrix}$ as $t \to \infty$

Markov Chain Monte Carlo

goal: efficiently sample from a desired probability distribution on a huge + complicated space

Why?

- TCS: approximate counting
- Stats + ML: Bayesian inference (posterior sampling)
- Statistical physics: estimate physical properties

Example: given an undirected graph $G = (V, E)$ count the # spanning forests of G

$\left(F \subseteq E \text{ s.t. } G = (V, F) \text{ is acyclic} \right)$

- NP-hard to compute exactly (even "#P-complete")
- We'll see poly time alg for # spanning trees (forests of size n-1)

Fact: can sample a near-uniform spanning forest in time poly$(n, 1/\varepsilon)$

$\Updownarrow$

can approximately count spanning forests to within factor $(1 \pm \varepsilon)$ in time poly$(n, 1/\varepsilon)$

(idea for $\Downarrow$: 1) pick an edge e

2) by repeated sampling, estimate

$\hat{p}_e \approx \Pr\left[\text{random spanning forest contains } e\right]$

3a) if $\hat{p}_e < \frac{1}{2}$, recursively estimate

$\hat{Z}_{G - \{e\}} \approx$ # spanning forests in $G \setminus \{e\}$

output $\dfrac{\hat{Z}_{G-\{e\}}}{1 - \hat{p}_e} = \hat{Z}_G$

spanning forests in G that don't contain e

3b) else, merge endpoints of e to get G', recursively estimate $\hat{Z}_{G'}$, output

$\dfrac{\hat{Z}_{G'}}{\hat{p}_e}$

spanning forests in G that contain e

Q: how to sample a near-uniform spanning forest?

Metropolis-Hastings Markov Chain

- State space $\mathcal{F}$ = spanning forests of $G=(V,E)$

  N = # states = as large as n^n

- Transitions: from forest $F \in \mathcal{F}$
  - pick a uniform random edge $e \in E$
  - let $F' = \begin{cases} F \triangle \{e\} & \text{if } F \triangle \{e\} \in \mathcal{F} \\ F & \text{o.w.} \end{cases}$

- RW on an $|E|$-regular graph on N vertices!

- Goal: mixing time poly(n) = poly($\log N$)

  i.e. $\omega_2 \leq 1 - 1/\text{poly}(n)$

  "rapidly mixing MC"

  Proven last year! using spectral theory of high-dimensional expanders

  [Anari - Liu - Oveis Gharan, STOC 2019]

Other Examples

- (Weighted) Perfect Matchings in Bipartite graphs — "Dimer model"
- (Weighted) Matchings — "Monomer-Dimer model"
- (Weighted) Independent Sets — "Hardcore Gas Model"
- (Weighted) Colorings — "Antiferromagnetic Potts Model"

Take-away messages on Markov Chains

- Estimating mixing time of r.w.'s on undirected graphs is equivalent (up to log factor) to bounding e-values
- Directed graphs more complicated
  - → can't read off stationary dist from degrees
  - → mixing time can be bounded by e-values/singular values (w_π) but not characterized by them
  - → mixing time can be exponentially large

Graph Coloring

Let G be a nonempty simple undirected graph

$\chi(G) =$ min # colors to color vertices s.t.
no adjacent vertices have same color

NP-hard to compute or even distinguish $\chi(G)=3$ vs. 4
or to approx. to within a factor $n^{1-\varepsilon}$

$\mu_1 \geq \mu_2 \geq \mu_3 \geq \cdots \geq \mu_n$ e-values of adj mx M

Claim: $\mu_n < 0$ if G nonempty

Pf: Because G is simple $\sum_{j=1}^{n} \mu_j = Tr(M) = 0$

$|\mu_n| \leq \mu_1$

Thm: $\underbrace{\left\lceil \frac{\mu_1}{-\mu_n} \right\rceil}_{=1 \text{ iff } G \text{ bipartite}} + 1 \leq \chi(G) \leq \lfloor \mu_1 \rfloor + 1$

Hoffman   Wilf

G bipartite $\Leftrightarrow G$ 2-colorable

K_n: $n \leq \chi(G) \leq n$ $\mu_1 = n-1$, $\mu_j = -1$ for $j > 1$

C_n: $2 \leq \chi(G) \leq 3$ n even
$3 \leq \chi(G) \leq 3$ n odd
$\mu_1 = 2$

Exercise: Calculate $\chi(G)$ and the bounds for $G = K_n$ (complete graph w/o self-loops) and $G = C_n$ (undirected n-cycle)

Proof of Hoffman:

If $k = \chi(G)$, can write ...

$$M = \begin{pmatrix} 0 & M_{12} & \cdots & M_{1k} \\ M_{21} & 0 & \ddots & \vdots \\ \vdots & & \ddots & M_{k,k-1} \\ M_{k1} & \cdots & M_{k,k-1} & 0 \end{pmatrix}$$

Lemma: $\sum_{j=1}^{k} \lambda_{max}(M_{jj}) \geq \lambda_{max}(M) + (k-1)\lambda_{min}(M)$

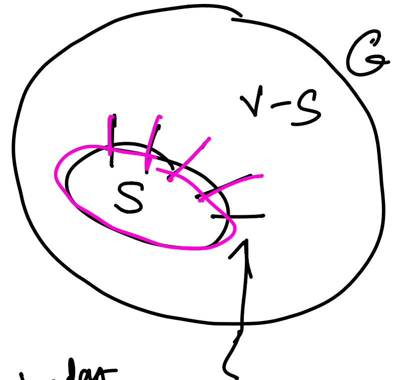
$$\begin{array}{ccc} \| & & \| \\ 0 & & \geq \mu_1 + (k-1)\mu_n \end{array}$$

$$\Rightarrow \quad k \geq 1 + \frac{\mu_1}{-\mu_n}.$$

Isoperimetry + Conductance

Let G be undirected

How well is S connected to $V-S$? for $|S| \le 1/2$



isoperimetric ratio

$$\theta(S) = \frac{|\partial S|}{|S|}$$

boundary

$$\partial S \overset{def}{=} \{(a,b) \in E : a \in S, b \notin S\}$$

$$\in \frac{|\partial S|}{n \cdot s(1-s)} \cdot [1, 2]$$

when $s \le 1/2$

$$s = \frac{|S|}{|V|}$$

Conductance

$$\phi(S) = \frac{|w(\partial S)|}{d(S)}$$

$$w(\partial S) = \sum_{e \in \partial S} w(e)$$

$$d(S) = \sum_{a \in S} d(a)$$

makes sense for digraphs

$$= \Pr\left[\begin{matrix}\text{r.w. started at } \pi|_S \\ \text{leaves } S \text{ in one step}\end{matrix}\right]$$

$\pi|_S$ = stationary conditional on S

$$\in \frac{\Pr\left[\begin{matrix}\text{r.w. started at } \pi \\ \text{has 1st vertex in } S \text{ and 2nd vertex in } V-S\end{matrix}\right]}{\pi(S) \cdot \pi(V-S)} \cdot [1, 2]$$

when $\pi(S) \le 1/2$

$$\pi(S) = \sum_{a \in S} \pi(a)$$

When G d-regular and unweighted, $\phi(S) = \dfrac{\theta(S)}{d}$

$$\Theta(G) \stackrel{\text{def}}{=} \min_{S : |S| \le n/2} \Theta(S)$$ isoperimetric number of G

$$\phi(G) \stackrel{\text{def}}{=} \min_{S : d(S) \le d(V)/2} \phi(S)$$ Conductance of G

If $|S| = n/2$

$x_S = (\underbrace{\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}}_{n/2}, -\tfrac{1}{2}, -\tfrac{1}{2}, \ldots -\tfrac{1}{2})$

$s = \frac{|S|}{|V|}$

Thm: 1) $\Theta(G) \ge \lambda_2/2$

2) $\phi(G) \ge \nu_2/2$

Proof idea

1)

$$\bar{\Theta}(S) = \frac{|\partial S|}{n \cdot s \cdot (1-s)} = \frac{x_S^T L x_S}{x_S^T x_S}$$

where x_S = component of 1_S orthogonal to $\vec{1}$ $= 1_S - s\vec{1}$

$$\min_S \bar{\Theta}(S) = \min_S \frac{x_S^T L x_S}{x_S^T x_S}$$ integer quadratic program $\to$ NP-hard

$$\ge \min_{x \perp \vec{1}} \frac{x^T L x}{x^T x}$$ convex relaxation – poly time (eigenvalue)

$$= \lambda_2$$

2)
$$\frac{w(\partial S)/d(V)}{\left(\frac{d(S)}{d(V)}\right)\left(\frac{d(S-V)}{d(V)}\right)} = \frac{y_S^T L y_S}{y_S^T D y_S}$$

where y_S = component of 1_S orthogonal to $\vec{d}$

$$= \frac{z_S^T N z_S}{z_S^T z_S}$$

where $z_S = D^{1/2} y_S \perp \vec{d}^{1/2}$

$$\min_{\text{over } S} \geq \min_{z \perp \vec{d}^{1/2}} \frac{z^T N z}{z^T z}$$

$$= \nu_2 .$$