Announcements

- start recording
- scribe: work w/other scribe to produce one set of notes
- My OH: Mon 12:30–1:30, Thu 9–10
- TF hybrid section/OH: Mon 2–3, Wed 2–3, Wed 5–6, Thu 5:30–6:30
- PS 1 posted, due Fri 9/25
- If Zoom goes down, check Piazza
- Join Theory Seminar mailing list

Agenda

1) Cayley graphs wrap-up
2) Perron-Frobenius for symmetric matrices
3) Random walks on undirected graphs

1) Cayley Graph Wrap-Up

$(G,+)$ a finite group, $W: G \to \mathbb{R}^{\geq 0}$

$\mathrm{Cay}(G, W)$ = digraph w/ vertex set G + edge weights $w(a,b) = W(b-a)$

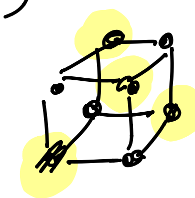
(when $W = 1_S$ for $S \subseteq G$, then $\mathrm{Cay}(G,W) = \mathrm{Cay}(G,S)$ w/ edges $\{(a, a+s) : s \in S\}$)

Thm: When G is abelian ($\forall a,b \; a+b = b+a$), then all of the graphs $\mathrm{Cay}(G,W)$ have a common orthogonal basis of complex eigenvectors, given by the homomorphisms $\chi : G \to S^1 \subseteq \mathbb{C}^*$ "Fourier basis"

+ Adj. mx of $\mathrm{Cay}(G,W)$ has e-values $\hat{W}_\chi = \sum_{s \in G} W(s)\chi(s)^*$ "Fourier transform" $= W\chi^*$

Example 1: Boolean hypercube $\quad G = \underbrace{\mathbb{Z}_2^d}_{\substack{\{0,1\}^d \\ \text{w/ addition mod 2}}} \quad S = \{e_1, e_2, \dots, e_d\}$

(i.e. $W = 1_S$)



Fourier basis : for each $r \in \mathbb{Z}_2^d$

$$\chi_r(x) = (-1)^{\langle r, x\rangle} = (-1)^{\langle r, x\rangle \bmod 2}$$

$$\hat{W}_r = \sum_{s \in \mathbb{Z}_2^d} 1_S(s) \cdot (-1)^{\langle r, s\rangle}$$

$$= \sum_{i=1}^{d} (-1)^{\langle r, e_i\rangle} = \sum_{i=1}^{d} (-1)^{r_i}$$

$$= \#(0\text{'s in } r) - \#(1\text{'s in } r) = d - 2\underbrace{|r|}_{\text{Hamming weight}}$$

$L = D - M = dI - M$

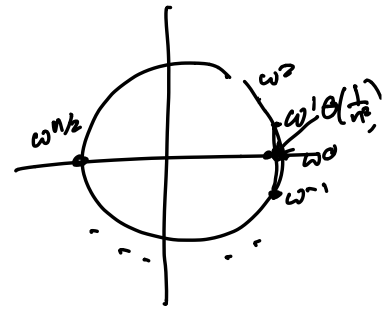
$\lvert r\rvert$	# e-values	e-value of M	e-value of L	e-value of W	e-value of N
0	1	d	0	1	0
1	d	$d-2$	2	$1-2/d$	$2/d$
2	$\binom{d}{2}$	$d-4$	4	$1-4/d$	$4/d$
3	$\binom{d}{3}$	$d-6$	6	$1-6/d$	⋮
⋮	⋮	⋮ 2	⋮	$2/d$	⋮
$d/2$	$\binom{d}{d/2}$	0	d	0	⋮
⋮	⋮	-2	$d+2$	$-2/d$	⋮
		⋮			⋮
$d-1$	d	$-(d-2)$	⋮		⋮
d	1	$-d$	$2d$	-1	2

MD^{-1}

Example 2: directed n-cycle $G=\mathbb{Z}_n$, $S=\{1\}$

Fourier basis: for $r \in \mathbb{Z}_n$

$$\chi_r(x) = \omega^{rx} \quad \text{for } \omega = e^{2\pi i/n}$$



ω^r - E-values? $\underset{r=0}{\omega^0}, \underset{r=1}{\omega^1}, \ldots, \omega^{n-1}$

undirected $S=\{\pm 1\}$ $\omega^r + \omega^{-r}$ $\cos(2\pi i r/n)$

Example 3: Noisy Hypercube $G=\mathbb{Z}_2^d$

$$W(s) = p^{|s|}\cdot(1-p)^{d-|s|}$$

E-values? Do $d=1$ first.

$$\hat{W}_r = \sum_{s\in\mathbb{Z}_2} W(s)\,\chi_r(s)^*$$

$$= \mathop{\mathbb{E}}_{s\sim Ber(p)^d}\left[(-1)^{\langle r,s\rangle}\right] = \mathop{\mathbb{E}}_{s\sim Ber(p)^d}\left[\prod_{i=1}^{d}(-1)^{r_i s_i}\right]$$

$$= \prod_{i=1}^{d} \underbrace{\mathop{\mathbb{E}}_{b\sim Ber(p)}\left[(-1)^{r_i b}\right]}_{1 \text{ if } r=0,\ (1-p)\cdot 1 + p\cdot(-1)}$$

$$= (1-2p)^{|r|} = \begin{cases} \approx 1-2p\cdot|r| & \text{when } |r| \ll \frac{1}{p} \\ \approx \exp(-k) & \text{when } |r| \geq \frac{k}{p} \end{cases}$$

Why Cayley graphs?

- use algebra to understand graph properties (e.g. eigenvalues) → or vice versa!
- highly symmetric ("vertex transitive")
- compact description of huge graphs (the set S)
- explicit constructions of large useful graphs (e.g. "expander graphs")
- connections to other useful combinatorial objects (e.g. error-correcting codes)

Abelian Cayley graphs

- easier to analyze (Fourier analysis)
- capture specific graphs of interest
- BUT have inherent limitations compared to non-abelian Cayley graphs (e.g. degree vs. expansion/λ_2/diameter)

Perron-Frobenius Thm for Symmetric Matrices

Let M = symmetric, nonnegative real matrix (e.g. adjacency or ~~...~~ normalized adj mx)
w/corresponding graph G (i.e. $E = \{(a,b): M(a,b) > 0\}$.
Eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ + Assume G connected

Then 1) $\exists$ strictly positive e-vector v_1 w/ e-value μ_1
(and the only nonneg. e-vectors are multiples of v_1)

2) $\mu_1 > \mu_2$ (cf. $\lambda_2 > 0 = \lambda_1$ iff G connected)

3) $\mu_1 \geq -\mu_n$ with equality iff G bipartite
Moreover, if G bipartite then $\mu_{n-i} = -\mu_{i+1}$ for $i = 0, \ldots, n-1$

Proof: 1) Let v be any e-vector w/ e-value μ_1

Non-negative e-vector: let $v_1 = |v|$ componentwise

$$\mu_1 = \frac{v^T M v}{v^T v} \leq \frac{v_1^T M v_1}{v_1^T v_1} \leq \mu_1 \Rightarrow v_1 \text{ e-vector of e-value } \mu_1$$

Strict positivity: assume for contradiction $v_1(a) = 0$ for some a
$\exists\, b$ s.t. $v_1(b) > 0$. $\boxed{r = \text{length of a path from } b \text{ to } a}$
consider $(M^r v_1)(a) > 0$

$= \mu_1^r v_1(a)$

positive value on b "propagates" to a ~~at~~ e-vector

2) omitted

3) apply #2 to M^2 - e-values $\mu_1^2, \mu_2^2, \ldots, \mu_n^2 \geq 0$

$\xrightarrow{\text{missing steps}} \mu_1^2 \geq \mu_i^2$ for $i = 2, \ldots, n$

if $\mu_1^2 = \mu_n^2$ then G^2 must be disconnected $\Rightarrow G$ bipartite

Random Walks on Undirected Graphs

G undirected, weighted. $W = MD^{-1}$ $\qquad$ $n = \#$ vertices

p_0 = initial probability distribution on vertices $\in \mathbb{R}^n$

p_t = prob dist. after t steps of r.w.

$= W^t p_0$

How to analyze?

Claim: W has a basis of e-vectors $\phi_1, \ldots, \phi_n$

w/ e-values $\omega_1 \geq \omega_2 \geq \omega_3 \geq \cdots \geq \omega_n$

Pf: $W = D^{1/2} A D^{-1/2}$. $\qquad A = D^{-1/2} M D^{-1/2}$ symmetric

orthonormal basis of e-vectors $\psi_1, \ldots, \psi_n$

$\phi_i = D^{1/2} \psi_i$

Write $p_0 = c_1\phi_1 + c_2\phi_2 + \cdots + c_n\phi_n$

$W^t p_0 = c_1\omega_1^t\phi_1 + c_2\omega_2^t\phi_2 + \cdots + c_n\omega_n^t\phi_n$

Will be dominated by eigenvalue(s) of largest magnitude

Thm: For start vertex a, end vertex b

$$|p_t(b) - \pi(b)| \leq \sqrt{\frac{d(b)}{d(a)}} \cdot \underline{\left(\max_{i>1} |\omega_i|\right)^t}$$

$\Rightarrow$ to get to within $\underbrace{\text{total variation distance}}_{\frac{1}{2}\,\ell_1 \text{ distance}}$ ε of π

suffices to have $\sqrt{\frac{d_{max}}{d_{min}}}\left(\max_{i>1} |\omega_i|\right)^t \leq \varepsilon$

how to ensure < 1?

Exercise: bound mixing time of random walk on hypercube

using eigenvalues

and using a combinatorial argument.