Announcements

- start recording
- scribe: work w/other scribe to produce one set of notes
- TFs: Chi-Ning Chou, Vinh-Kha Le, Alec Sun, Santhoshini Velusamy
- whiteboard live sync on Google drive (link on Piazza)
- My OH: Mon 12:30–1:30, Thu 9–10
- Sections: Mon 2–3, Wed 5–6
- TF OH: Wed 2–3, Thu 5–6
- PS0 due tomorrow 5pm, PS1 to be posted over weekend
- if Zoom goes down, check Piazza

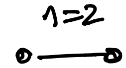
Agenda

1) correction from last time
2) diagonalization over $\mathbb{C}$
3) groups
4) general Cayley digraphs and examples
5) Fourier eigenbasis
6) eigenvalues

1) Correction from Last Time

- complete graph w/ ~~self-loops~~ has $\lambda_2 = \lambda_3 = \cdots = \lambda_n = n = \cancel{d}\ d+1$
- largest possible value of $\frac{\lambda_2}{d} = \frac{n}{n-1}$

$n=2$



Proof: Let L be Laplacian of any d-regular graph

$$d \cdot n \geq Tr(L) = \sum_{i=1}^{n} \lambda_i \geq (n-1) \cdot \lambda_2$$

$0 = \lambda_1 \leq \lambda_2 \leq \cdots$

- largest possible value of $\frac{\lambda_n}{d} = 2$

achieved in any bipartite graph

2) Diagonalization over $\mathbb{C}$

Spectral Thm over $\mathbb{R}$: Let $M \in \mathbb{R}^{n \times n}$ ($n \times n$ real matrices)

TFAE (the following are equivalent):

a) $\exists$ orthonormal basis $v_1, \ldots, v_n \in \mathbb{R}^n$ of real eigenvectors of M

b) $M = V \Lambda V^T$ for orthogonal matrix $V \in \mathbb{R}^{n \times n}$ and diagonal $\Lambda \in \mathbb{R}^{n \times n}$

c) M is symmetric

What are complex analogues?

- for $z = x + iy \in \mathbb{C}$, $z^* = x - iy$
- for $v \in \mathbb{C}^n$, $M \in \mathbb{C}^{n \times n}$, v^*, M^* = conjugate transpose
- $\|v\| = \sqrt{v^* v}$, $\langle v, w \rangle = v^* w$
- orthonormal basis of $\mathbb{C}^n$: $v_1, \ldots, v_n \in \mathbb{C}^n$ s.t. $v_i^* v_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{o.w.} \end{cases}$
- unitary matrix $V \in \mathbb{C}^{n \times n}$: $V^* V = I$
- M is Hermitian if $M^* = M$

Thm: For $M \in \mathbb{C}^{n\times n}$, TFAE

a) $\exists$ orthonormal basis $v_1, \dots, v_n \in \mathbb{C}^n$ of complex e-vectors of M

b) $M = V^* \Lambda V$ for unitary V and diagonal Λ

c) M normal : $\underbrace{M^* M = M M^*}_{}$

more general than Hermitian or symmetric even for real M

2) Groups

Def: a group is a set Γ w/ a binary operation $\circ$ s.t.

a) $\forall\, x, y, z \in \Gamma \quad (x \circ y) \circ z = x \circ (y \circ z)$ [associativity]

b) $\exists\, e \in \Gamma \quad \forall x \in \Gamma \quad e \circ x = x \circ e = x$ [identity]

c) $\forall x \in \Gamma\ \exists y \in \Gamma \quad x \circ y = y \circ x = e$ [inverses]

A group $(\Gamma, \circ)$ is abelian if $\forall x, y \in \Gamma \quad x \circ y = y \circ x$ [commutativity]

Examples of groups

- $(\mathbb{R}, +)$ $\quad$ - $(\mathbb{R}^{n\times n}, +)$ $\quad$ - $(\mathbb{Z}, +)$
- $\mathbb{Z}_n = (\{0, \dots, n-1\}, + \bmod n) \cong \mathbb{Z}/n\mathbb{Z} =$ integers "modulo" equivalence relation $a \equiv b$ if $n \mid a-b$

  ($n-1 \equiv -1$)
- $(\{0,1\}^d, \text{bitwise } \oplus) \cong (\mathbb{Z}/2\mathbb{Z})^d$

e.g. $d=3$

000
001
010
100
011
101
110
111

$001 \oplus 100 = 101$

- Every finite abelian group is $\cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$
- $\mathbb{R}^* = (\mathbb{R} - \{0\}, \times)$, $\mathbb{C}^* = (\mathbb{C} - \{0\}, \times)$
- $S^1 = (\{z \in \mathbb{C} : |z| = 1\}, \times)$ inverse of z $= z^*$
- ($n \times n$ invertible real matrices, $\times$)
- ($n \times n$ unitary complex matrices, $\times$)

] nonabelian when $n > 1$

3) <u>general Cayley digraphs</u>

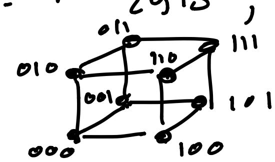
For a finite group $(\Gamma, +)$ and $S \subseteq \Gamma$,

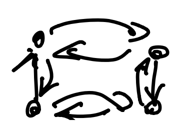
$\mathrm{Cay}(\Gamma, S)$ is the $|S|$-regular digraph with
- vertex set Γ
- edges $\{(x, x+s) : s \in S\}$

(connected iff S "generates" ~~the~~ Γ)

<u>Examples:</u>

1) <u>Hypercube</u>: $\Gamma = \{0,1\}^d$, $S = \{e_1, \ldots, e_d\}$

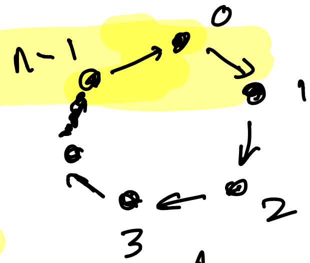




note ~~the~~ $s + s = 0$ in Γ

2) $\Gamma = \mathbb{Z}_n$, $S = \{1\}$:

abelian



directed

$S = \{\pm 1\}$
$= \{1, n-1\}$
to get undirected n-cycle

3) $S = \Gamma$: complete graph w/ self-loops

Weighted Cayley digraphs:

given $w_0 : \Gamma \to \mathbb{R}^{\geq 0}$ $\quad$ $(S = \{s : w_0(s) > 0\})$

$\mathrm{Cay}(\Gamma, w_0)$ has

- vertex set Γ
- edge weights $w(a,b) = w_0(b-a)$
- d-regular w/ $d = \sum_{s \in \Gamma} w_0(s)$

(so $\mathrm{Cay}(\Gamma, S) = \mathrm{Cay}(\Gamma, 1_S)$)

indicator fn S

Example: noisy hypercube NH_p for $p \in [0,1]$

for $s \in \{0,1\}^d$ $\quad w(s) = p^{|s|} \cdot (1-p)^{d-|s|}$ $\quad |s| = \#$ 1's in s

$= \Pr_{\mathrm{Ber}(p)^d}[s]$ $\quad$ = "Hamming weight"

$NH_p \approx \mathrm{Cay}\left(\{0,1\}^d, \left\{\begin{matrix}\text{strings of Hamming} \\ \text{weight} \approx p \cdot d\end{matrix}\right\}\right)$

Let Γ be a finite abelian group, $S \subseteq \Gamma$

M = adjacency matrix of $\mathrm{Cay}(\Gamma, S)$

W = rw. matrix $= M/d$ $\qquad d = |S|$

$\mathcal{L}$ = normalized Laplacian $= I - W$

Claim: M (and hence $W, \mathcal{L}$) is normal.

Pf: $M^* M = M^T M = \mathrm{Cay}(\Gamma, \{s-t : s,t \in S\})$ even for nonabelian, with multiplicity

$\| $ abelian

$M M^* = M M^T = \mathrm{Cay}(\Gamma, \{-s+t : s,t \in S\})$

in general: for any S, T $\mathrm{Cay}(\Gamma, S)$ & $\mathrm{Cay}(\Gamma, T)$ commute

common diagonalization

4) Fourier eigenbasis

Thm: Assume Γ abelian. An orthogonal set of complex eigenvectors for $\mathrm{Cay}(\Gamma, S)$ (ie for M, $\mathcal{L}$ and W) is given by the set of characters $\chi : \Gamma \to \mathbb{C}$

$$\chi : \Gamma \to S^1 \subseteq \mathbb{C}$$

s.t. $\forall x,y$ $\chi(x+y) = \chi(x)\cdot\chi(y)$

"homomorphisms"

Examples

1) characters for $\mathbb{Z}_n = \{\{0,\dots,n-1\}, + \bmod n)$

$$\chi_r(x) = e^{(2\pi i r x)/n} \qquad r \in \{0,\dots,n-1\}$$

e.g. $n=4$

$$\begin{array}{c|ccccc}
 & \chi(0) & \chi(1) & \chi(2) & \chi(3) & \cancel{\chi(4)} \\
r=0 & (1 & 1 & 1 & 1) & \\
r=1 & (1 & i & -1 & -i) & \\
r=2 & (1 & -1 & 1 & -1) & \\
r=3 & (1 & -i & -1 & i) &
\end{array}$$

2) characters for hypercube $\cong \mathbb{Z}_2^d$

for $r \in \{0,1\}^d$

$$\chi_r(\underline{x}) = (-1)^{r_1 x_1}(-1)^{r_2 x_2}\cdots(-1)^{r_n x_n}$$

$$= (-1)^{\langle r, x\rangle}$$

5) Calculating eigenvalues

$$(M\underline{\chi})(a) = \sum_{b\in\Gamma} M(a,b)\chi(b)$$

$$= \sum_{b\in\Gamma} w(b,a)\chi(b)$$

$$= \sum_{b\in\Gamma} w_0(a-b)\chi(b)$$

$$= \sum_{s\in\Gamma} w_0(s)\chi(a-s)$$

homomorphism

$$= \sum_{s\in\Gamma} w_0(s)\chi(a)\chi(-s)$$

$$= \chi(a)\sum_{s\in\Gamma} w_0(s)\cdot\chi(s)^*$$

unweighted

$$= \chi(a)\cdot\sum_{s\in S}\chi(s)^*$$

$$= \chi(a)\cdot\lambda_\chi$$

Thm addition:

The eigenvalue corresponding to eigenvector χ is $\sum_{s\in\Gamma} w_0(s)\cdot\chi(s)^*$

Exercise: 1) work out n e-values of directed n-cycle (Trev: undirected n-cycle)

2) work out 2^d e-values of Noisy hypercube NH_ρ (Trev: ordinary hypercube)