Announcements

- start recording
- scribe: work w/other scribe to produce one set of notes
- My OH: Mon 12:30-1:30, Thu 9-10
- TF section/OH: Mon 2-3, Tue 8-9, Wed 5-6, Thu 5:30-6:30
- PS 4 posted. Project proposal due Sun 11/8, problems due TUE 11/17
- If Zoom goes down, check Piazza
- Sync whiteboard
- Jamboard link in chat

Agenda

- Recap: resistor networks
- Series + Parallel Composition
- Schur Complements

Resistor Networks

G undirected, weighted, connected graph.

View each edge as a resistor w/ resistance $r_e = \frac{1}{w_e}$

Given voltages $v : V \to \mathbb{R}$

Ohm's Law $(V = IR)$ says:

signed

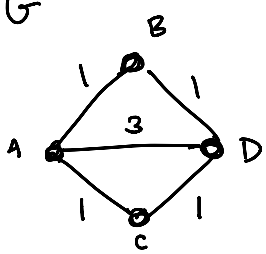
$$\text{current on edge } \overrightarrow{(a,b)} = \frac{v(a) - v(b)}{r_{(a,b)}} = w_{(a,b)}\,(v(a) - v(b))$$

$$\vec{i} = \text{Net current from } a = \sum_b w_{(a,b)}\,(v(a) - v(b))$$

$$= (Lv)(a)$$

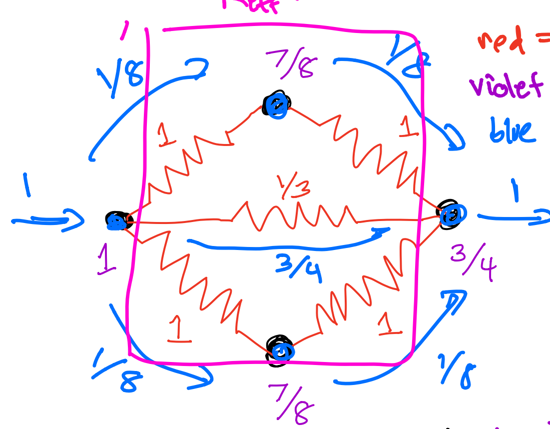
Example:

G



$$L = \begin{array}{c} \begin{pmatrix} 5 & -1 & -1 & -3 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -3 & -1 & -1 & 5 \end{pmatrix} \\ \begin{matrix} A & B & C & D \end{matrix} \end{array}$$

$R_{eff}(A,D) = \frac{1}{4}$



red = resistance
violet = voltage
blue = current

$$\delta_A - \delta_D = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 & -1 & -1 & -3 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -3 & -1 & -1 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 7/8 \\ 7/8 \\ 3/4 \end{pmatrix}$$

Flow conservation: net current at each a must equal zero.

$(Lv)(a) = 0 \iff (Dv)(a) = (Mv)(a)$

$\iff v(a) = \frac{1}{d(a)} \sum_{b \sim a} w_{a,b} v(b)$

"v harmonic at a"

$((I-W)\pi)(a) = 0 \iff \pi(a) = (MD^{-1}\pi)(a)$

$\iff \pi(a) = \sum_{b \sim a} \frac{w_{b,a} \pi(b)}{d(b)}$

$\Rightarrow$ $i_{ext}=$ $Lv \neq 0$ requires external currents

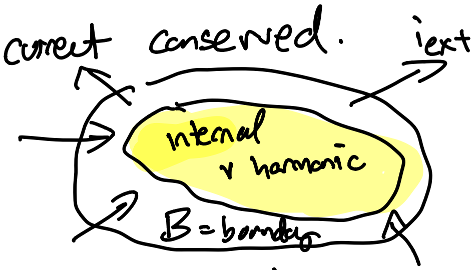
only solutions to $Lv=0$ are $v = c\vec{1}$ $\Rightarrow$ no current on any edges

$i_{ext} = Lv$ to realize voltages v

$i_{ext}(a)$ = external current entering a

Note: $i_{ext} \perp \vec{1}$, so external current conserved.

$\overset{\parallel}{Lv}$



Given any $i_{ext} \in \mathrm{Im}(L)$, $v = L^+ i_{ext}$ are induced voltages

$\overset{\parallel}{\vec{1}^{\perp}}$ PS3

up to adding mult. of $\vec{1}$

Effective Resistance between a and b:

$$R_{eff}(a,b) = \left[v(a) - v(b) \text{ when } i_{ext} = \delta_a - \delta_b \right]$$

$$= (L^+ i_{ext})(a) - (L^+ i_{ext})(b)$$

$$= (\delta_a - \delta_b)^T L^+ (\delta_a - \delta_b)$$

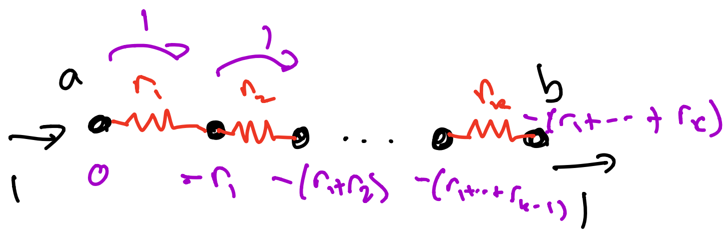
motivation to calculate L^+ efficiently

$$= \| L^{+/2} \delta_a - L^{+/2} \delta_b \|^2$$

$$\sqrt{R_{eff}(a,c)} \le \sqrt{R_{eff}(a,b)} + \sqrt{R_{eff}(b,c)}$$

Exercise for Breakouts

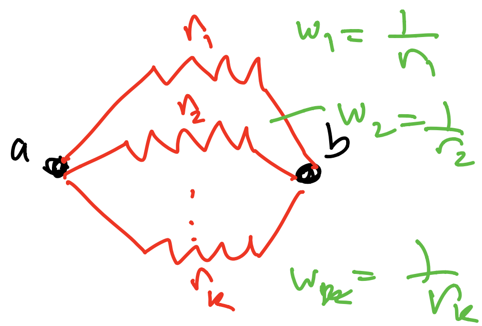
Derive effective resistances between a and b in the following networks by finding voltages v inducing current $i_{ext} = \delta_a - \delta_b$ (i.e. $Lv = \delta_a - \delta_b$)



"series"

$$-r_1 = \frac{1}{\left(\frac{1}{r_1}+\frac{1}{r_2}\right)}\cdot\left(\frac{0}{r_1} + \frac{-(r_1+r_2)}{r_2}\right)$$

$$R_{eff}(a,b) = r_1 + \dots + r_k$$



"parallel"

$$W_{(a,b)} = w_1 + \dots + w_k$$

$$\frac{1}{R_{eff}(a,b)} = \frac{1}{r_1} + \dots + \frac{1}{r_k}$$

$$R_{eff}(a,b) = \frac{1}{\frac{1}{r_1} + \dots + \frac{1}{r_k}}$$

Schur Complements

Given $v(B)$ and constraint that v harmonic on $V-B$, find $i_{ext}(B)$.

$$x_S^T L(S,S) x_S = \begin{pmatrix} x_S \\ 0 \end{pmatrix}^T L \begin{pmatrix} x_S \\ 0 \end{pmatrix} > 0$$

because $\begin{pmatrix} x_S \\ 0 \end{pmatrix} \notin \mathrm{Span}(\mathbb{1})$

Thm: 1) $i_{ext}(B) = L_B v(B)$

for $L_B = L(B,B) - L(B,S) L(S,S)^{-1} L(S,B)$, $S = V - B$

why invertible? $x^T L(S,S) x > 0$

"Schur Complement" of L with respect to S or "on B"

where $L = \begin{pmatrix} L(S,S) & L(S,B) \\ L(B,S) & L(B,B) \end{pmatrix}$

2) L_B = matrix obtained from L by using Gaussian elimination to eliminate S

3) L_B is a Laplacian

$$L(S,S)v(S) + L(S,B)v(B) = \vec{0}_S$$

Proof:

1) given $v(B)$, define $v(S) = -L(S,S)^{-1}L(S,B)v(B)$

then
$$\begin{pmatrix} L(S,S) & L(S,B) \\ L(B,S) & L(B,B) \end{pmatrix}\begin{pmatrix} v(S) \\ v(B) \end{pmatrix} = \begin{pmatrix} 0 \\ L_B v(B) \end{pmatrix}$$

$$L_B v(B) = -L(B,S)L(S,S)^{-1}L(S,B)v(B) + L(B,B)v(B)$$

2) Proof for $S = \{1\}$, $B = \{2, \ldots, n\}$

$$\left(\begin{array}{c|c} L(1,1) & L(1,B) \\ \hline L(B,1) & L(B,B) \end{array}\right)\left(\begin{array}{c} v(1) \\ \hline v(B) \end{array}\right) = \left(\begin{array}{c} 0 \\ \hline i_{ext}(B) \end{array}\right)$$

$\Downarrow$ row ops

$$\left(\begin{array}{c|c} L(1,1) & L(1,B) \\ \hline 0 & L(B,B) - \dfrac{L(B,1)L(1,B)}{L(1,1)} \end{array}\right)\left(\begin{array}{c} v(1) \\ \hline v(B) \end{array}\right) = \left(\begin{array}{c} 0 \\ \hline i_{ext}(B) \end{array}\right)$$

$\parallel$

Schur Complement wrt row 1

3) L_B an (undirected) Laplacian:

- symmetric
- off-diagonal non-positive
- row + column sums $= 0$

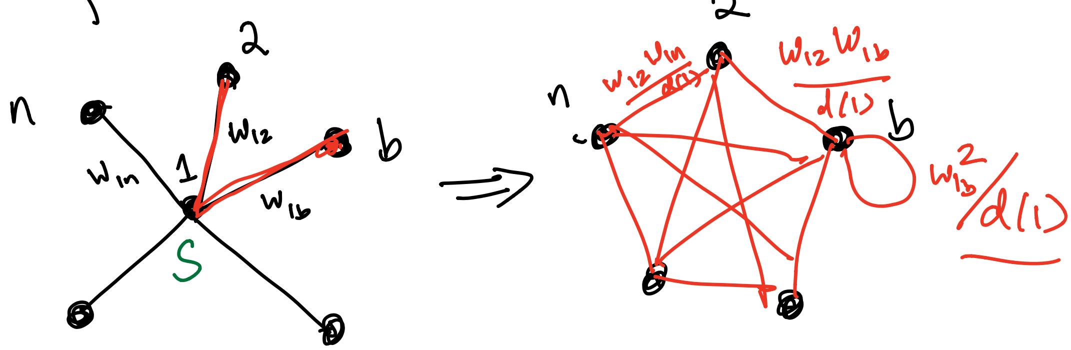
$\Leftrightarrow$ Laplacian of a weighted undirected graph

$$\times \frac{w_{1b}}{d(1)} \qquad + $$

$$\begin{pmatrix} d(1) & -w_{12} & \cdots & -w_{1b} & \cdots & -w_{1n} \\ \vdots & \ddots & & & & \vdots \\ -w_{b1} & -w_{b2} & \cdots & d(b) & \cdots & -w_{bn} \\ \vdots & & & \vdots & & \vdots \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} d(1) & -w_{12} & & -w_{1b} & \cdots & -w_{1n} \\ \vdots & & \ddots & & & \vdots \\ 0 & -w_{2b} - \frac{w_{12}w_{1b}}{d(1)} & & d(b) - \frac{w_{1b}^2}{d(1)} & \cdots & -w_{bn} - \frac{w_{1b}w_{1n}}{d(1)} \\ \vdots & & & \vdots & & \vdots \end{pmatrix}$$



star

clique

Random Walk Interpretation of Schur Complements

$$L_{rw} = I - W = \begin{pmatrix} I_S - W_{SS} & -W_{SB} \\ -W_{BS} & I_B - W_{BB} \end{pmatrix}$$

Schur Complement wrt S

$$= L_{rw}(B,B) - L_{rw}(B,S)\, L_{rw}(S,S)^{-1} L_{rw}(S,B)$$

$$= I_B - W_{BB} - (-W_{BS})(I_S - W_{SS})^{-1}(-W_{SB})$$

$$= I_B - \Big(W_{BB} + W_{BS}\big(\underbrace{I_S + W_{SS} + W_{SS}^2 + W_{SS}^3 + \cdots}_{\text{walk any number of steps in } S}\big) \cdot W_{SB} \Big)$$

random walk of 1 step staying in B

return to B

walk any number of steps in S

go from B to S in 1 step

RW matrix stationary dist will be π conditioned on being in B

Thm: R_{eff} is a metric, i.e.

$$(\delta_a - \delta_b) L^+ (\delta_a - \delta_b) > 0$$

$$L^+(\delta_a - \delta_b)$$

✓ ① $R_{eff}(a,b) \geq 0$ for all a,b with equality iff $a=b$

✓ ② $R_{eff}(a,b) = R_{eff}(b,a)$ for all a,b

③ $R_{eff}(a,c) \leq R_{eff}(a,b) + R_{eff}(b,c)$ for all a,b,c

Proof: Use Schur Complements to eliminate all vertices except a, b, c

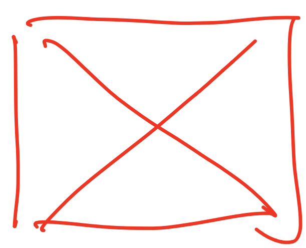
$\Rightarrow$ preserves $R_{eff}(a,b)$, $R_{eff}(a,c)$, $R_{eff}(b,c)$

By series+parallel composition

$$R_{eff}(a,b) = \frac{1}{\frac{1}{r_{ab}} + \frac{1}{r_{bc}+r_{ac}}}$$

same for $R_{eff}(b,c)$ and $R_{eff}(a,c)$

and check



can't be built by series+parallel composition