Announcements

- start recording
- scribe: work w/other scribe to produce one set of notes
- My OH: Mon 12:30–1:30, Thu 9–10
- TF section/OH: Mon 2–3, Tue 8–9, Wed 5–6, Thu 5:30–6:30
- Fixing Problem 3.3 now extra credit
- If Zoom goes down, check Piazza
- Sync whiteboard
- Let us know in PS3 submission if still looking for partners
- Jamboard link in chat

Agenda

- Explicit Constructions of Expanders
- Undirected S-T Connectivity in Logspace
- Resistor Networks

Recap: Operations on Graphs

(n,d,γ) graph : n vertices, degree d, spectral expansion $\geq \gamma$

Squaring: $(n,d,\gamma) \mapsto (n, d^2, 2\gamma-\gamma^2)$ $\qquad (n,d,1-\omega) \mapsto (n, d^2, 1-\omega^2)$

Tensoring: $(n,d,\gamma) \mapsto (n^2, d^2, \gamma)$ $\qquad (n,d,1-\omega) \mapsto (n^2, d^2, 1-\omega)$

Zig-Zag: $(n_1,d_1,\gamma_1) \circledz (d_1,d_2,\gamma_2) \mapsto (n_1 d_1, d_2^2, \gamma_1\gamma_2^2)$ $\qquad (n,d_1,1-\omega_1) \circledz (d_1,d_2,1-\omega_2) \mapsto (n_1 d_1, d_2^2, 1-(\omega_1+2\omega_2))$

Constructing Expanders

Let $H = (d^4, d, 7/8)$ graph (PS 3)

$$\begin{cases} G_1 = H^2 & d^2\text{-regular} \\ G_{t+1} = G_t^2 \circledz H & d^2\text{-regular} \quad n_{t+1} = d^4 \cdot n_t = d^{4(t+1)} \end{cases}$$

$$\omega_{t+1} \leq \omega_t^2 + 2\cdot\left(\tfrac{1}{8}\right) \leq \tfrac{1}{2}$$

induction

$$\text{Time}(G_{t+1}) = 2\,\text{Time}(G_t) + O(\log n_t)$$

to compute neighbors

"edge-rotation map"

$$= \Theta(t \cdot 2^t)$$

$$= n_t^{\Theta(1)}$$ not good enough

Soln: use tensoring too

S-T Connectivity

Given $G=(V,E)$, $s,t \in V$, is there a path from s to t?

- Directed G: · time and space $O(n)$ (DFS)
  - · or space $O(\log^2 n)$ (M^n via repeated squaring)

Boolean matrix powers

$$M^n(s,t) = \bigvee_{a \in V} M^{n/2}(s,a)\, M^{n/2}(a,t)$$

depth $= O(\log n)$

stack space per level $= O(\log n)$

recursive alg: for each $a \in V$ recursively check if $M^{n/2}(s,a) \neq 0$ and if $M^{n/2}(a,t) \neq 0$

for $M^n(s,t) \neq 0$

time $2^{O(\log^2 n)} = n^{\log n}$

- Undirected G: · in randomized space $O(\log n)$ (PS2, 1979)

Here: deterministic space $O(\log n)$ [Reingold 2004]

s_0, t_0 $\quad G_0 =$ d-regular, aperiodic modification of G

s_{t+1}, t_{t+1} $\quad G_{t+1} = G_t^2 ⓩ H \qquad H$ a $(d^4, d, 3/4)$-graph

$\gamma_0 \geq 1/\mathrm{poly}(n) \leftarrow$ on component containing s

$\gamma_{t+1} \geq (2\gamma_t - \gamma_t^2) \cdot \left(\frac{3}{4}\right)^2 \approx \frac{18}{16} \cdot \gamma_t$ for small γ_t

$\gamma_{O(\log n)} \geq \Omega(1) \implies G_{O(\log n)}$ a constant-degree expander.

$\mathrm{Space}(G_{t+1}) = \mathrm{Space}(G_t) + O(1)$. → try all paths of length $O(\log n)$

describe a path of length ℓ in a degree d graph $= \ell \cdot \log d$ bits

Digraphs w/ poly(n) mixing time : RL-complete

i.e. deterministic logspace alg. for S-T connectivity on digraphs w/ poly(n) mixing time $\Rightarrow$ RL = L.

open!

Resistor Networks

G undirected, weighted, connected graph.

View each edge as a resistor w/ resistance $r_e = \frac{1}{w_e}$

Given voltages $v : V \to \mathbb{R}$

Ohm's Law ($V = IR$) says:

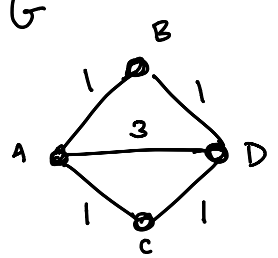
$$\text{current on edge } \overrightarrow{(a,b)} = \frac{v(a) - v(b)}{r_{(a,b)}} = w_{(a,b)} \underbrace{\left(v(a) - v(b)\right)}_{}^{\text{signed}}$$

$$\vec{i} = \text{Net current from } a = \sum_b w_{(a,b)} \left(v(a) - v(b)\right)$$

$$= (Lv)(a)$$

Example:

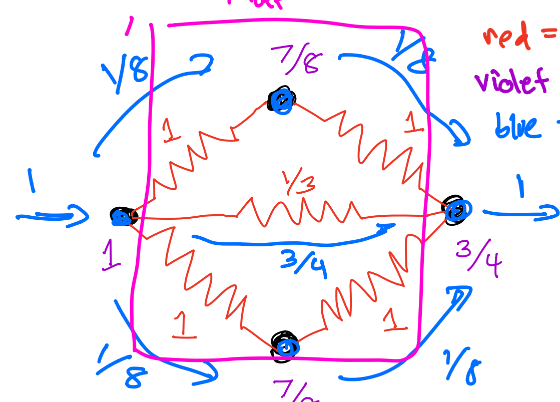
G



$$\Longrightarrow$$

$R_{eff}(A,D) = \frac{1}{4}$

red = resistance
violet = voltage
blue = current



$$L = \begin{array}{c} \begin{pmatrix} 5 & -1 & -1 & -3 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -3 & -1 & -1 & 5 \end{pmatrix} \\ \begin{matrix} A & \;\; B & \;\; C & \;\; D \end{matrix} \end{array}$$

$$\delta_A - \delta_D = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 & -1 & -1 & -3 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -3 & -1 & -1 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 7/8 \\ 7/8 \\ 3/4 \end{pmatrix}$$

Flow conservation: net current at each a must equal zero.

$\Rightarrow$ $\underbrace{Lv \neq 0}$ requires external currents

$i_{ext} =$

$i_{ext} = Lv$ to realize voltages v

$i_{ext}(a)$ = external current entering a

only solutions to $Lv = 0$ are $v = c\vec{1}$ $\Rightarrow$ no current on any edges

Note: $\underbrace{i_{ext}}_{\substack{\parallel \\ Lv}} \perp \vec{1}$, so external current conserved.

Given any $i_{ext} \in \underset{\substack{\parallel \\ \vec{1}^{\perp}}}{Im(L)}$, $\underbrace{v = L^+ i_{ext}}_{PS3}$ are $\underline{\text{induced voltages}}$

up to adding mult. of $\vec{1}$

Effective Resistance between a and b:

$$R_{eff}(a,b) = [v(a) - v(b) \text{ when } i_{ext} = \delta_a - \delta_b]$$

$$= (L^+ i_{ext})(a) - (L^+ i_{ext})(b)$$

$$= (\delta_a - \delta_b)^T L^+ (\delta_a - \delta_b)$$

motivation to calculate L^+ efficiently

$$= \| L^{+/2}\delta_a - L^{+/2}\delta_b \|^2$$

if M is psd w/ evalues $\lambda_1, \dots, \lambda_n \geq 0$ e-vectors $v_1, \dots, v_n$ $\Longrightarrow$ $M^{1/2}$ has evectors $v_1, \dots, v_n$ and evalues $\lambda_1^{1/2}, \dots, \lambda_n^{1/2}$

Thm:

$$\frac{1}{R_{eff}(a,b)} = \min_{\substack{x \in \mathbb{R}^V \text{ s.t.} \\ x(a) = 1 \\ x(b) = 0}} \left(x^T L x\right)$$

Claim: x is unique minimizer

- partial derivatives of $x^T L x$ are zero at x (except for a and b^{th} coordinates) — flow constraints
- strict convexity of $x^T L x$

$f(x) = x^T L x$

Proof sketch: Let $i_{ext} = \delta_a - \delta_b$

$$v = L^+ i_{ext}$$

$$x = \frac{v - v(b)\mathbf{1}}{R_{eff}(a,b)}$$

$x(b) = 0$

$$x(a) = \frac{v(a) - v(b)}{R_{eff}(a,b)} = 1$$

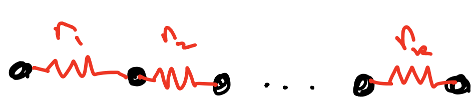
Plug in this x and $x^T L x = \frac{1}{R_{eff}(a,b)}$

Cor: If we decrease resistances (= increase edge weights), effective resistances can't increase.

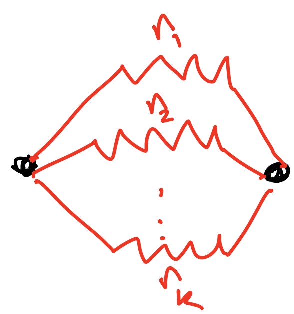
(Rayleigh's Monotonicity Principle)

Exercise for Breakouts

Derive effective resistances between a and b in the following networks by finding voltages v inducing current $i_{ext} = \delta_a - \delta_b$ (i.e. $Lv = \delta_a - \delta_b$)



"series"



"parallel"