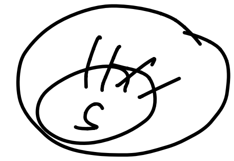
Announcements

- start recording
- scribe: work w/other scribe to produce one set of notes
- My OH: Mon 12:30–1:30, Thu 9–10
- TF hybrid section/OH: Mon 2–3, Wed 2–3, Wed 5–6, Thu 5:30–6:30
- PS 2 posted, including final project ideas problem
- If Zoom goes down, check Piazza
- sync whiteboard
- gather after class: link in chat
- jamboard link in chat

Agenda

- Recap/finish: proof of Cheeger
- Higher-order Cheeger

Recap: $\phi(S) \overset{\text{def}}{=} \frac{w(\delta S)}{d(S)} = \frac{\sum_{a \in S, b \notin S} w_{ab}}{\sum_{a \in S} d(a)}$



$$\phi(G) \overset{\text{def}}{=} \min_{S : d(S) \le d(V)/2} \phi(S) \quad \text{conductance of } G$$

Thm: $$\frac{\nu_2}{2} \le \phi(G) \le \sqrt{2\nu_2}$$

Cheeger's Inequality

• in a connected unweighted graph $\phi(G) > \frac{1}{nd}$

Proof of Cheeger's Inequality

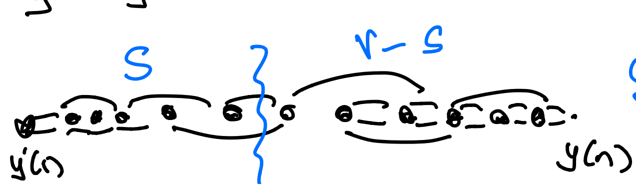
Goal: Given $\vec{y} \perp \vec{d}$ s.t. $\frac{y^T L y}{y^T D y} \le \rho$ (e.g. $\vec{y} = D^{1/2} \psi_2$, $\rho = \nu_2 \le 2\phi(G)$)

"round" $\vec{y}$ to obtain a set S (i.e. a $\{0,1\}$ vector 1_S)

s.t. $\frac{w(\delta S)}{\min\{d(S), d(V-S)\}} \le \sqrt{2\rho}$ $\le 2\sqrt{2\phi(G)}$

if $\rho = \nu_2$

Sort y $y(1) \le \cdots \le y(n)$



will take $S = \{a : y(a) \le t\}$ for some t

In fact, will define a distribution on thresholds t

s.t. $\underline{\mathbb{E}_\tau\left[w(\partial S_\tau)\right]} \le \underline{\sqrt{2\rho}} \cdot \underline{\mathbb{E}_\tau\left[\min\{d(S_\tau), d(V-S_\tau)\}\right]}$

Step 1: center $\vec{y}$

- $\vec{z} = \underline{\vec{y} - s\vec{1}}$ for appropriate s

  so that $\sum_{a: z(a)<0} d(a) \le \frac{d(V)}{2}$ and $\sum_{a: z(a)>0} d(a) \le \frac{d(V)}{2}$

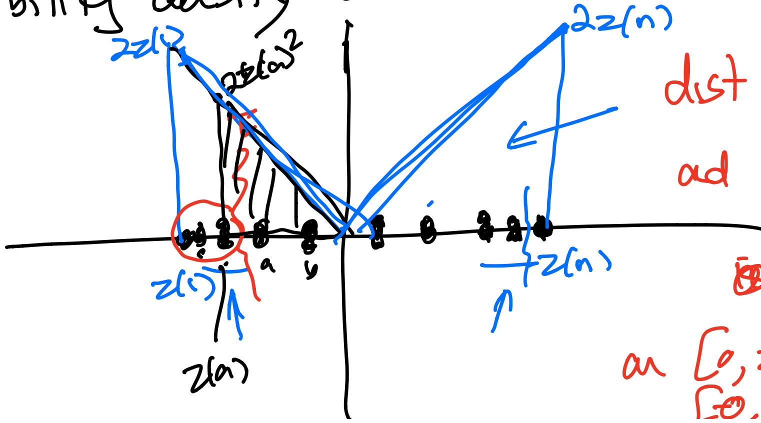
- $\dfrac{\vec{z}^T L \vec{z}}{\vec{z}^T D \vec{z}} \le \rho$

- Assume WLOG $\underline{z(1)^2 + z(n)^2 = 1}$

- $S_\tau \stackrel{\text{def}}{=} \{a : z(a) \le \tau\}$

Step 2: distribution on τ

- Probability density at t $= 2|t|$



dist of $t^2 | t>0$

and dist of $t^2 | t<0$

~~are~~ uniform

on $[0, z(n)^2]$ and

$[0, z(1)^2]$

Claim: $$\mathop{\mathbb{E}}_{\tau}\left[\min\{d(S_\tau),\, d(V-S_\tau)\}\right] = z^T D z$$

Pf: $$\text{LHS} = \sum_a d(a)\cdot \Pr\left[a \text{ is in "smaller" of } S_\tau \text{ and } V-S_\tau\right]$$

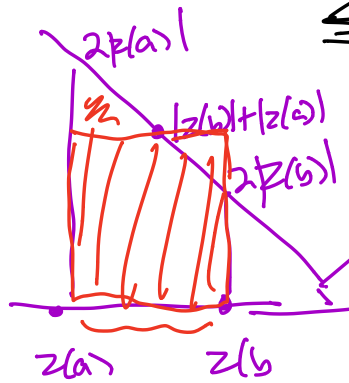
$$\overset{\text{centering}}{=} \sum_a d(a)\cdot \Pr\left[\tau \text{ is between } z(a) \text{ and } 0\right]$$

$$= \sum_a d(a)\cdot z(a)^2$$

$$= z^T D z$$

Claim: $$\mathop{\mathbb{E}}_{\tau}\left[w(\partial S_\tau)\right] \le \sqrt{2}\sqrt{z^T L z}\cdot\sqrt{z^T D z}$$

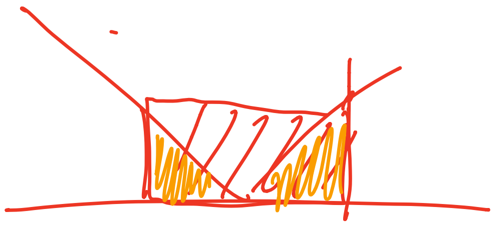
Pf: $$\text{LHS} = \sum_{(a,b)\in E} w_{ab}\cdot \Pr\left[\tau \text{ is between } z(a) \text{ and } z(b)\right]$$



$$\le \sum_{(a,b)\in E} w_{ab}\cdot \underbrace{|z(b)-z(a)|\cdot\left(|z(b)|+|z(a)|\right)}_{u_{ab}\cdot v_{ab}}$$

$$\overset{\text{C-S}}{\le} \sqrt{\sum_{(a,b)\in E} w_{ab}\cdot \underbrace{(z(b)-z(a))^2}_{u_{ab}^2}}\cdot\sqrt{\sum_{(a,b)\in E} w_{ab}\cdot \underbrace{\left(|z(b)|+|z(a)|\right)^2}_{v_{ab}^2}}$$

$$\le \sqrt{z^T L z}\cdot\sqrt{2 z^T D z}$$



$$u_{ab} = \sqrt{w_{ab}} \cdot |z(b) - z(c)|$$

$$v_{ab} = \sqrt{w_{ab}} \cdot \left(|z(b)| + |z(c)|\right)$$

Cauchy-Schwarz Inequality

v1: for every two vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$

$$|\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\| \cdot \|\vec{v}\|$$

v2: for all $u_1, \ldots, u_n, v_1, \ldots, v_n \in \mathbb{R}$

$$\left(\sum_{i=1}^{n} u_i v_i\right)^2 \leq \left(\sum_{i=1}^{n} u_i^2\right) \cdot \left(\sum_{i=1}^{n} v_i^2\right)$$

G undirected

Higher-Order Cheeger

Fact: $\nu_k = 0 \iff G$ has at least k connected components

Pf: 1) If G has exactly k connected components

$$N = \begin{pmatrix} N_1 & 0 & 0 & 0 \\ 0 & N_2 & 0 & 0 \\ 0 & 0 & \ddots & \\ & & & N_k \end{pmatrix}$$

$$\frac{x^T N x}{x^T x} = \frac{y^T L y^T}{y^T D y}$$

$y = D^{1/2} x$

N_i = normalized laplacians for components

eigenvalues(N) = union w/ mult. of e-values of N_i

0 has mult. k

2) $\nu_k = 0 \iff \dim(\ker(N)) \geq k \iff \dim(\ker(L)) \geq k$

$\ker(L) = \mathrm{span}(1_{S_1}, \dots, 1_{S_k})$

S_i = connected components

Higher-Order Cheeger:

$\nu_k \approx 0 \iff G$ "close" to having $\geq k$ components

$\exists$ disjoint $S_1 \dots S_k \subseteq V$

s.t. $\phi(S_1), \dots, \phi(S_k)$ "small"

$$\phi_k(G) \stackrel{\text{def}}{=} \min_{\substack{S_1, \dots, S_k \\ \text{disjoint}}} \max \{\phi(S_1), \dots, \phi(S_k)\}$$

Thm: $\frac{\nu_k}{2} \leq \phi_k(G) \leq \text{poly}(k) \cdot \sqrt{\nu_k}$

Exercise for breakouts: try to reconstruct proof that $\frac{\nu_k}{2} \leq \phi_k(G)$

Let $\rho = \phi_k(G)$

Have disjoint ~~sets~~ $S_1, \ldots, S_k$

s.t. $\phi(S_i) \leq \rho$ for each i

$$\phi(S_i) = \frac{1_{S_i}^T L 1_{S_i}}{1_{S_i}^T D 1_{S_i}} \leq \rho$$

Let $S = \text{span}(1_{S_1}, \ldots, 1_{S_k})$

By C-F

$$\nu_k \leq \max_{\substack{x \in S \\ x \neq 0}} \frac{x^T L x}{x^T D x} \overset{?}{\leq} 2\rho$$

uses $1_{S_1}, \ldots, 1_{S_k}$ are disjointly supported

Proof idea for $\phi_k(G) \le \text{poly}(k) \cdot \sqrt{\nu_k}$

Assume G d-regular and unweighted for simplicity

Let $F_1, \ldots, F_k : V \to \mathbb{R}$ be $\in \mathbb{R}^V$

k orthonormal vectors s.t.

$$F_i^T N F_i = \underbrace{\frac{F_i^T L F_i}{d}}_{\rho_i} \le \rho \quad \left(\begin{array}{c}\text{e.g.} \\ \rho = \nu_k\end{array}\right)$$

Define $F : V \to \mathbb{R}^k$ $\quad F(a) = \begin{pmatrix} F_1(a) \\ \vdots \\ F_k(a) \end{pmatrix}$ cf. Hall's graph drawing

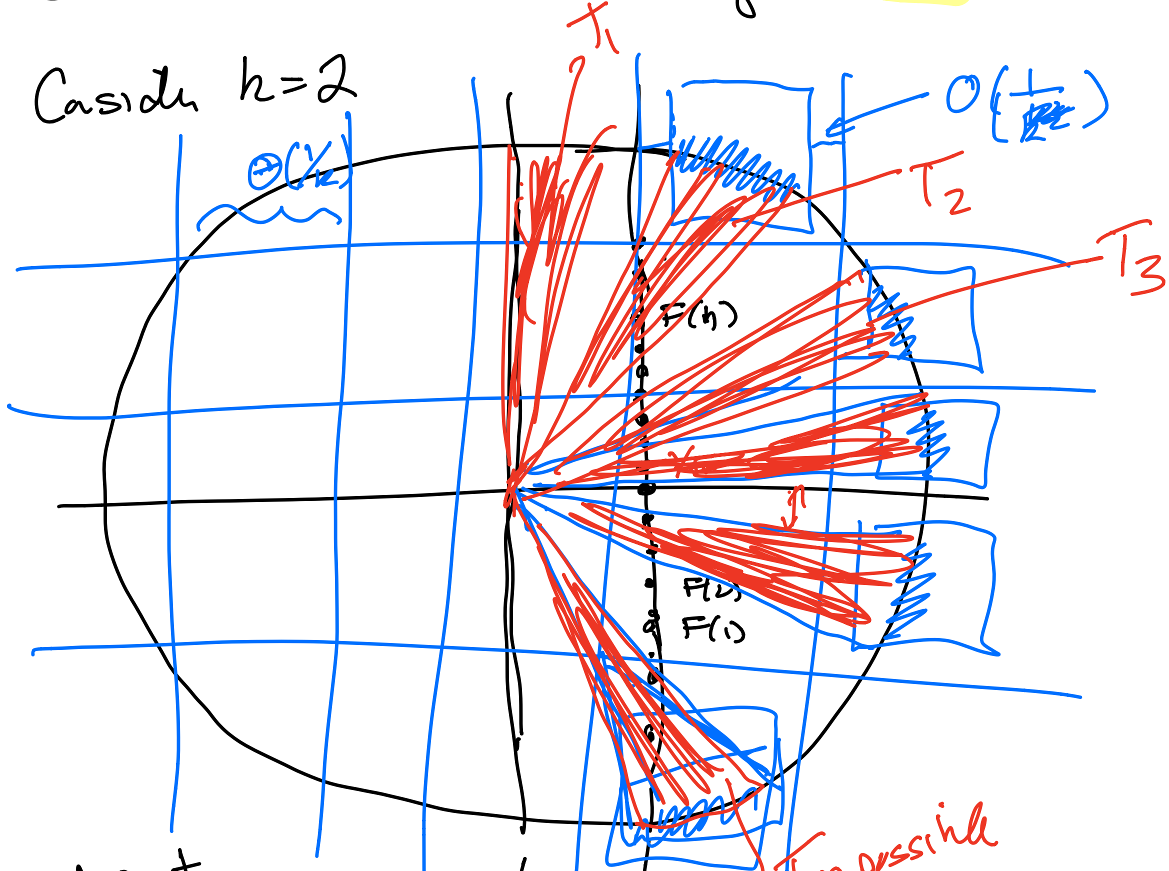
Know: $$\frac{\sum_{(a,b)\in E} \|F(a) - F(b)\|^2}{d} \overset{?}{\le} \sum_{i=1}^{k} \rho_i = k \cdot \rho$$

Also: $\sum_{a \in V} \|F(a)\|^2 = k$

How to "round" and get sets $S_1, \ldots, S_k$?

Can't round each F_i separately. (why?)

Consider $k=2$



$\Rightarrow$ disjoint sets $T_1, \ldots, T_m \subseteq V$ s.t.

Impossible but $m >> k$

- $\sum_{a \in \bigcup T_i} \|F(a)\|^2 \geq k \cdot (1 - O(1/k))$

- $\forall i \ \sum_{a \in T_i} \|F(a)\|^2 \leq 1 + O(1/k)$

- $\forall i \neq j, a \in T_i, b \in T_j \quad \left\| \frac{F(a)}{\|F(a)\|} - \frac{F(b)}{\|F(b)\|} \right\| \geq \frac{1}{\text{poly}(k)}$

$$\| \qquad \sqrt{2 - 2\cos\left(\text{angle between } F(a) \text{ and } F(b)\right)}$$

$\Rightarrow$ disjoint sets $A_1, \dots, A_k$ s.t.

- $\forall i \quad \sum_{a \in A_i} \|F(a)\|^2 \geq 1/2$
- $\forall i \neq j \quad a \in A_i, b \in A_j \quad \left\| \frac{F(a)}{\|F(a)\|} - \frac{F(b)}{\|F(b)\|} \right\| \geq \frac{1}{\mathrm{poly}(k)}$

$$y_i(a) = \begin{cases} \|F(a)\| & \text{if } a \in A_i \\ \text{interpolate o.w.} & \\ 0 & \text{if } a \text{ is "far" from } A_i \end{cases}$$

"localization"

$\Rightarrow$ disjointly supported nonnegative vectors $y_1, \dots, y_k$

s.t. $$\frac{y_i^T N y_i}{y_i^T y_i} \leq \mathrm{poly}(k) \cdot \frac{F^T N F}{F^T F}$$

$$= \mathrm{poly}(k) \cdot \rho$$

$\Rightarrow$ disjoint sets $S_1, \dots, S_k$ s.t.

Cheeger-like rounding

$$\phi(S_i) \leq \sqrt{2 \frac{y_i^T N y_i}{y_i^T y_i}} \leq \mathrm{poly}(k) \cdot \sqrt{\rho}$$