CS 229r Spectral Graph Theory in Computer Science, Lecture 3	Sep 10, 2020

Lecture 3

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1 Recap / Correction from Last Time

Complete graphs without self-loops

1. The Laplacian matrix of a complete graph has eigenvalues $\lambda_2 = \lambda_3 = \cdots = \lambda_n = n = d + 1$
2. Largest possible value of $\lambda_2/d = n/(n-1)$.

   Proof. Let L be the Laplacian of any $d-$regular graph.

   $$d \cdot n = \mathrm{Tr}(L) = \sum_{i=1}^{n} \lambda_i \geq (n-1)\lambda_2$$
   $$\implies \frac{n}{n-1} \geq \frac{\lambda_2}{d}$$

   □

   This means that *asymptotically*, $\lambda_2/d \to 1$, but it can take a higher value (it is 2 for $n = 2$).

3. Largest possible value of $\lambda_n/d = 2$ (we will see this in more detail in a later class; it is achieved in any connected bipartite graph).

2 Diagonalization on $\mathbb{C}$

Theorem 1 (Spectral Theorem on $\mathbb{R}$)**.** *Let $M \in \mathbb{R}^{n\times n}$ ($n\times n$ real matrices), then the following are equivalent (TFAE):*

1. *There exists an orthonormal basis $v_1, \ldots, v_n \in \mathbb{R}^n$ of **real** eigenvectors of M*
2. *$M = V\Lambda V^\top$ for **orthogonal** matrix $V \in \mathbb{R}^{n\times n}$ and diagonal $\Lambda \in \mathbb{R}^{n\times n}$*
3. *M is symmetric*

But what are the complex analogues of the above objects?

- For $z = x + iy \in \mathbb{C}$, $z^* = x - iy$
- For $v \in \mathbb{C}^n, M \in \mathbb{C}^{n\times n}$, v^*, M^* are called *conjugate transposes*
- $\|v\| = \sqrt{v^* v}$ (norm), $\langle v, w\rangle = v^* w$ (inner product)
- Orthonormal basis of $\mathbb{C}^{n\times n}$: $v_1, \ldots, v_n \in \mathbb{C}^n$ s.t. $\langle v_i, v_j\rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$
- Unitary matrix $V \in \mathbb{C}^{n\times n}$: $V^* V = I$
- M is Hermitian if $M^* = M$

Theorem 2. *For $M \in \mathbb{C}^{n\times n}$, the following are equivalent:*

1. *There exists orthonormal basis $v_1, \ldots, v_n \in \mathbb{C}^n$ of **complex** eigenvectors of M*
2. *$M = V^*\Lambda V$ for unitary V and diagonal Λ*
3. *M is normal: $M^*M = MM^*$. Note: This is a more general condition than M being Hermitian or symmetric (even for real matrices).*

3 Groups

Definition 3 (Group)**.** *A group is a set Γ with a binary operation $\circ$ such that:*

1. *$\forall x, y, z \in \Gamma$, $(x \circ y) \circ z = x \circ (y \circ z)$ (associativity)*
2. *$\exists e \in \Gamma$ s.t. $\forall x \in \Gamma$, $e \circ x = x \circ e = x$ (indentity)*
3. *$\forall x \in \Gamma$, $\exists y \in \Gamma$, $x \circ y = y \circ x = e$ (inverses)*

Definition 4 (Abelian Group)**.** *A group $(\Gamma, \circ)$ is abelian if $\forall x, y \in \Gamma$: $x \circ y = y \circ x$ (commutativity).*

Examples of groups

1. $(\mathbb{R}, +)$ - addition on the set of real numbers
2. $(\mathbb{R}^{n\times n}, +)$ - addition of the set of real, square matrices
3. $(\mathbb{Z}, +)$ - addition of the set of integers
4. $\mathbb{Z}_n = (\{0, \ldots, n-1\}, + \mod n) \cong \mathbb{Z}/n\mathbb{Z}$ - integers "modulo" equivalence relation ($a \equiv b$ if $n|a-b$; '$\cong$': 'congruent to')
5. $(\{0,1\}^d, \text{bitwise} \bigoplus) \cong (\mathbb{Z}/n\mathbb{Z})^d$ - bitwise xor on the $d-$dimensional hypercube
6. Every finite abelian group is $\cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$
7. $\mathbb{R}^* = (\mathbb{R} - \{0\}, \times)$, $\mathbb{C}^* = (\mathbb{C} - \{0\}, \times)$
8. $S^1 = (\{z \in \mathbb{C} : |z| = 1\}, \times)$ (where inverse of $z = z^*$)
9. ($n \times n$ invertible real matrices, $\times$) - nonabelian when $n > 1$
10. ($n \times n$ unitary complex matrices, $\times$) - nonabelian when $n > 1$

4 General Cayley Digraphs

Definition 5 (Cayley Digraph)**.** *For a finite group $(\Gamma, +)$ and a subset $S \subseteq \Gamma$, Cay(Γ, S) is the $|S|-$regular digraph with*

- *vertex set: Γ*
- *edges: $\{(x, x+s) : s \in S\}$*

It is connected iff S "generates" Γ.

Examples

1. Hypercube: $\Gamma = \{0,1\}^d$, $S = \{e_1, \ldots, e_d\}$
2. Directed $n-$cycle: $\Gamma = \mathbb{Z}_n$ (which is abelian), $S = \{1\}$
3. Undirected $n-$cycle: $\Gamma = \mathbb{Z}_n$, $S = \{1, -1\} = \{1, n-1\}$
4. Complete graph (with self-loops): $S = \Gamma$

Weighted Cayley Digraphs

Definition 6 (Weighted Cayley Digraphs)**.** *Given a weight function on the group, $w_0 : \Gamma \to \mathbb{R}^{\geq 0}$, Cay$(\Gamma, w_0)$ has:*

- *vertex set:* Γ
- *edge weights:* $w(a, b) = w_0(b - a)$
- *$d-$regular with* $d = \sum_{s \in \Gamma} w_0(s)$

Note that this definition is more general than the previous one. More specifically, $\mathrm{Cay}(\Gamma, S) = \mathrm{Cay}(\Gamma, \mathbf{1}_S)$, where $\mathbf{1}_S$ is the indicator for set S.

Example: Noisy Hypercube, NH_p for $p \in [0,1]$

For $s \in \{0,1\}^d$:

$$\begin{aligned} w(s) &= p^{|s|} \cdot (1-p)^{d-|s|} \\ &= \Pr(s) = \mathrm{Bern}(p)^d \end{aligned}$$

where $|s|$ is the number of 1's in s (which is the 'Hamming weight' of s).

Finite abelian groups

Let Γ be a finite abelian group, $S \subseteq \Gamma$, then:

1. M is the adjacency matrix of $\mathrm{Cay}(\Gamma, S)$
2. $W = M/d$ is the random walk matrix, where $d = |S|$
3. $L = I - W$ is the normalized Laplacian

Claim 7. *M (and hence W, L) is normal*

Proof. We note that the M is a real matrix, and M^*M, MM^* have nice interpretations themselves (in terms of Cayley graphs):

$$\begin{aligned} M^*M = M^\top M &= \mathrm{Cay}(\Gamma, \{s - t : s, t \in S\}) \\ MM^* = MM^\top &= \mathrm{Cay}(\Gamma, \{-s + t : s, t \in S\}) \end{aligned}$$

where the sets $\{s - t : s, t \in S\}, \{-s + t : s, t \in S\}$ are with multiplicity. Because the group is abelian, we will have the two sets being the same. ∎

In general, for every S, T, $\mathrm{Cay}(\Gamma, S)$ and $\mathrm{Cay}(\Gamma, T)$ commute with each other (i.e. their adjacency matrices commute). This further implies that the adjacency matrices have a common diagonalization, with a common basis for all choices of S, T. The Fourier eigenbasis is that basis, and lets us get an explicit handle on what that basis is.

5 Fourier eigenbasis

Theorem 8. *Assume Γ to be abelian. An orthogonal set of **complex** eigenvectors for Cay(Γ, S) (for M, L, and W) is given by the set of characters $\chi : \Gamma \to \mathbb{C}$, where:*

$$\begin{aligned} &\chi : \Gamma \to S' \subseteq \mathbb{C} \\ s.t.\ \forall x, y: \quad &\chi(x+y) = \chi(x)\chi(y) \qquad \textit{(homomorphism)} \end{aligned}$$

Examples

1. Characters for $\mathbb{Z}_n = (\{0, \ldots, n-1\}, + \mod n)$: $\chi_r(x) = e^{2\pi i r x/n}$, $r \in \{0, \ldots, n-1\}$.

   For $n = 4$:

	$\chi(0)$	$\chi(1)$	$\chi(2)$	$\chi(3)$
$r = 0$	1	1	1	1
$r = 1$	1	i	-1	-i
$r = 2$	1	-1	1	-1
$r = 3$	1	-i	-1	i

2. Characters for hypercube $\cong \mathbb{Z}_2^d$:

   For $r \in \{0,1\}^d$:

$$\begin{aligned} \chi_r(x) &= (-1)^{r_1 x_1}(-1)^{r_2 x_2} \ldots (-1)^{r_n x_n} \\ &= (-1)^{\langle r, x \rangle} \end{aligned}$$

6 Calculating eigenvalues

$$\begin{aligned} (M\chi)(a) &= \sum_{b \in \Gamma} M(a,b)\chi(b) \\ &= \sum_{b \in \Gamma} w(b,a)\chi(b) \\ &= \sum_{b \in \Gamma} w_0(a-b)\chi(b) \\ &= \sum_{s \in \Gamma} w_0(s)\chi(a-s) \\ &= \sum_{s \in \Gamma} w_0(s)\chi(a)\chi(-s) && \text{(homomorphism)} \\ &= \chi(a)\sum_{s \in \Gamma} w_0(s)\chi(s)^* && \left(= \chi(a)\sum_{s \in \Gamma} \chi(s)^* \text{ for unweighted case}\right) \\ &= \chi(a)\lambda_\chi \end{aligned}$$

Here we find that the eigenvalue corresponding to the eigenvector χ is $\sum_{s \in \Gamma} w_0(s)\chi(s)^*$

7 Exercise for the next class

1. Work out the n eigenvalues of directed $n-$cycle (Trevisan does undirected $n-$cycles)
2. Work out the 2^d eigenvalues of the Noisy Hypercube, NH_p (Trevisan does ordinary hypercube)