CS120: Intro. to Algorithms and their Limitations	Prof. Salil Vadhan

Problem Set 5

Harvard SEAS - Fall 2021 *Due: Wed Oct. 20, 2021 (5pm)*

Your name:
Collaborators:
No. of late days used on previous psets:
No. of late days used after including this pset:

The purpose of this problem set is to gain familiarity with the implementation and behavior of Randomized Algorithms and with proving formal statements about Graph Algorithms.

1. (Randomized Algorithms in Practice)

   (a) Implement Randomized QuickSelect, filling in the template we have given you in the Github repository.

   (b) In the repository, we have given you datasets x_n of item-key pairs of varying sizes to experiment with. For each dataset x_n and any given number k, you will compare two ways of answering the k selection queries $[n/k], [2n/k], \ldots, [(k-1)n/k]$ on x_n, where $[\cdot]$ denotes rounding to the nearest integer:

      i. Running Randomized QuickSelect k times

      ii. Running MergeSort (provided in the repository) once and using the sorted array to answer the k queries

   Specifically, you will compare the *distribution* of runtimes of the two approaches for a given pair (n, k) by running each approach many times and creating density plots of the runtimes. The runtimes will vary because Randomized QuickSelect is randomized, and because of variance in the execution environment (e.g. what other processes are running on your computer during each execution).

   We have provided you with the code for plotting. Before plotting, you will need to implement MergeSortSelect, which extends MergeSort to answer k queries. Your goal is to use these experiments and the resulting density plots to propose a value for for $k*$ (as a function of n) at which you should switch over from Randomized QuickSelect to MergeSort. Do this by experimenting with the parameters for k (code is included to generate the appropriate queries once the ks are provided) and generate a plot for each experiment. Explain the rationale behind your choice, and submit a few density plots for each value of n to support your reasoning. (There is not one right answer, and it may depend on your particular implementation of QuickSelect.)

2. (Analyzing BFS) Let $G = (V, E)$ be a connected, undirected graph in which every vertex has degree 2. Consider running BFS on G from an arbitrary vertex v_0, let F_d be the frontier at the start of the d'th iteration, and $S_d = \bigcup_{i=0}^{d} F_i$ the set of vertices visited so far. Below, you will reason about how BFS behaves on G and use this to conclude that G must be a *cycle*.[1]

[1] That is, there is an ordering $u_0, u_1, \ldots, u_{\ell-1}$ of the vertices of G such that $E = \{\{u_0, u_1\}, \{u_1, u_2\}, \ldots, \{u_{\ell-2}, u_{\ell-1}\}, \{u_{\ell-1}, u_0\}\}$.

(a) Show that $|F_1| = 2$.

(b) Let d^* be the largest value of d such that $|F_d| = 2$. Prove by induction on $d = 1, \ldots, d^*$ that the two (shortest) paths of length d from v_0 to the vertices in F_d are disjoint except for their starting point v_0 and together contain exactly the vertices in the set S_d.

(c) Argue that either we have $|F_{d^*+1}| = 0$ or we have $|F_{d^*+1}| = 1$ and $|F_{d^++2}| = 0$.

(d) Show how to obtain a cycle C containing v_0 from the shortest paths found by BFS. Using the fact that BFS visits all vertices reachable from v_0, deduce that C includes all vertices and edges of G. That is, the entire graph G is a cycle.

3. **MOVED TO PSET 6** ~~(Rotating Paths) Suppose we are given k digraphs on the same vertex set, $G_0 = (V, E_0), G_1 = (V, E_1), \ldots, G_{k-1} = (V, E_{k-1})$. For vertices $s, t \in V$, an *rotating path* with respect to $G_0, \ldots, G_{k-1}$ from s to t is a sequence of vertices $v_0, v_1, \ldots, v_\ell$ such that $v_0 = s$, $v_\ell = t$, and $(v_i, v_{i+1}) \in E_{i \bmod k}$ for $i = 0, \ldots, \ell - 1$. That is, we are looking for paths that rotate between the digraphs $G_0, G_1, \ldots, G_{k-1}$ in the edges used.~~

   (a) ~~Show that the shortest k-rotating path (if one exists) from s to t can be found in time $O(kn + m_0 + \cdots + m_{k-1})$, where $n = |V|$ and $m_i = |E_i|$. (Hint: give a reduction to ordinary shortest paths by constructing from $G_0, \ldots, G_{k-1}$ a digraph G' on kn vertices with $m_0 + \cdots + m_{k-1}$ edges. Alternatively, modify BFS appropriately.)~~

   (b) ~~A group of three friends decides to play a new cooperative game. They rotate turns moving a shared single piece on an $n \times n$ grid. The piece starts in the lower-left corner, and their goal is to get the piece to the upper-right corner in as few turns as possible. Many of the spaces on the grid have visible bombs, so they cannot move their piece to those spaces. Each player is restricted in how they can move the piece. Player 0 can move it like a chess-rook (any number of spaces vertically or horizontally, provided it does not cross any bomb spaces). Player 2 can move it like a chess bishop (any number of spaces diagonally in any direction, provided it does not cross any bomb spaces). Player 3 can move it like a chess knight (move to any non-bomb space that is two steps away in a horizontal direction and one step away in a vertical direction or vice-versa). Using Part 3a, show that given the $n \times n$ game board (i.e. the locations of all the bomb spaces), they can find the quickest solution in time $O(n^3)$. (Hint: give a reduction, mapping the given grid to an appropriate instance $(G_0, G_1, \ldots, G_{k-1}, s, t)$ of Shortest k-Rotating Paths.)~~