Announcements

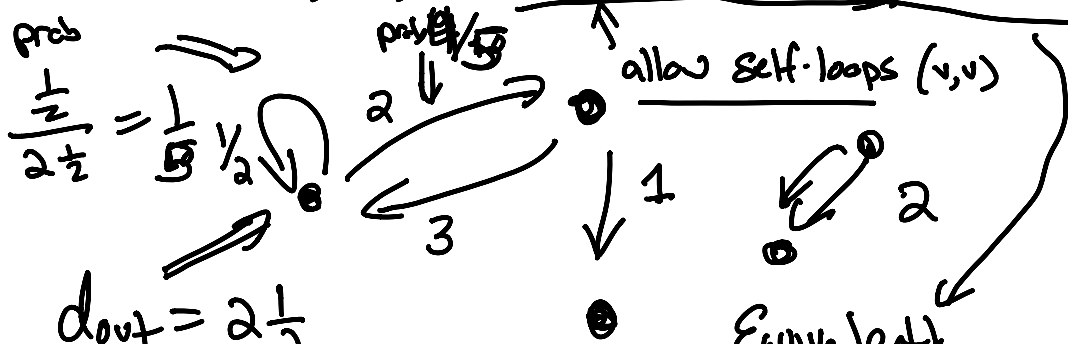
- include pronouns in Zoom name
- mute self when not speaking
- videos on, don't record other students or share class recordings
- TFs: Chi-Ning Chou, Alec Sun, Santhoshini Velusamy, + TBD
- scribe
- use Zoom raise hand
- start recording
- live sync of notes
- PS0 due Fri 9/11
- my OH Thurs 9-10am
- fill out section + OH poll ASAP
- use Piazza to find pset partners
- in case Zoom goes down, check Piazza

Graphs

most general for us = weighted digraph

$G = (V, E) \quad E \subseteq V \times V \quad w: E \to \mathbb{R}^+$

allow self-loops (v,v)



Equivalently

$w: V \times V \to \mathbb{R}^{\geq 0}$

$E = \{ e \in V \times V : w(e) > 0 \}$

Special cases

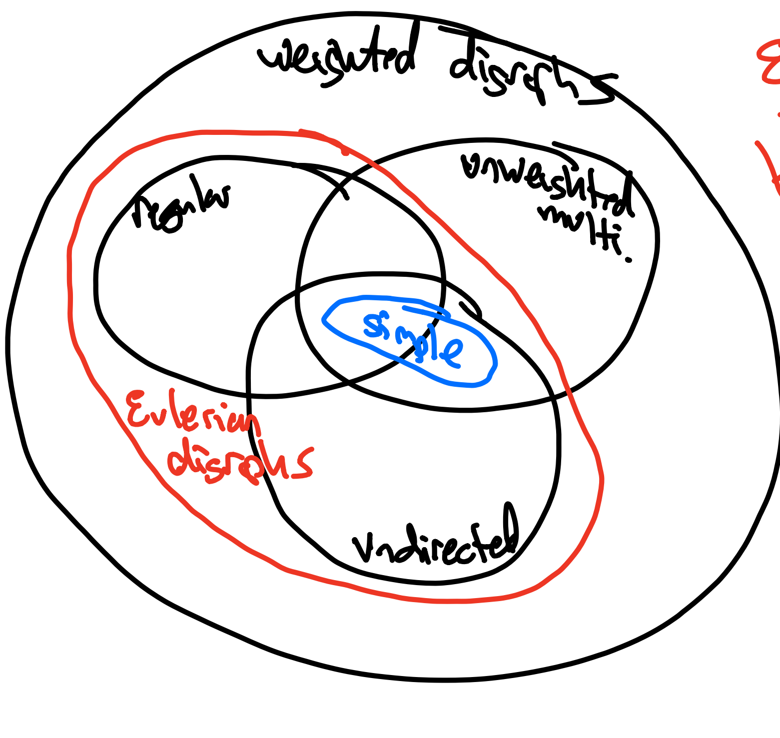
Spielman + TODAY

- undirected: $w(a,b) = w(b,a) \quad \forall a,b$
- unweighted multigraph: $w : V \times V \to \mathbb{N}$

  $w(a,b)$ = multiplicity of (a,b)
- regular: $\exists d \; \forall a \quad d_{out}(a) = d_{in}(a) = d$

$$d_{out}(a) \overset{def}{=} \sum_b w(a,b)$$

$$d_{in}(a) = \sum_b w(b,a)$$



Eulerian digraphs

$\forall a \quad d_{in}(a) = d_{out}(a)$

Simple graph

undirected

unweighted

no parallel edges

no self-loop

Matrices

M_G

- adjacency matrix $M(a,b) = w(b,a)$
- random-~~walk~~ matrix $W = M \cdot D_{out}^{-1}$
  diffusion

$$D_{out} = \begin{pmatrix} d_{out}(1) & & 0 \\ & \ddots & \\ 0 & & d_{out}(n) \end{pmatrix}$$

$$W(a,b) = \frac{w(b,a)}{d_{out}(b)}$$

$$D_{in} = \ldots$$

In Eulerin digraph$^{(*)}$, $D_{in} = D_{out} = D$

(*) Laplacian: $L = D - M$

RW Laplacian: $L_{rw} = I - W \overset{(*)}{=} \underline{L D^{-1}}$

(*) Normalized Laplacian: $N = I - D^{-1/2} M D^{-1/2}$

$$= D^{-1/2} L D^{-1/2}$$

Assume G Eulerian!

Matrix	e-vector	e-value	symmetric if G undirected?
L	$\vec{1}$	0	✓
$L_{rw} = I - W$	$\vec{d} = D\vec{1}$	0	not nec. when G irregular
W	$\vec{d}$	1	"
N	$\vec{d^{1/2}}$	0	✓
~~A~~ M	??		✓

Note: when G is d-regular all of these are equivalent to study

$$D = dI$$

$\vec{1}$ is e-vector of $\cancel{A}\ M = dI - L$ of e-~~vector~~ value d.

Spectral Thm: If M is a symmetric matrix, then $M = V \Lambda V^T$ for an orthogonal matrix V $\left(\begin{array}{c} VV^T = I \\ V^TV = I \\ \text{cols/rows orthonormal} \end{array} \right)$ and a diagonal matrix Λ.

$$M = \sum_{i=1}^{n} \lambda_i v_i v_i^T$$

λ_i: diag of Λ ; v_i: cols of V

equivalently $v_1, \dots, v_n$ are an orthonormal basis of e-vectors of M w/ e-values $\lambda_1, \dots, \lambda_n$

For asymmetric matrices, we have SVD

$M = U \Sigma_1 V^T$ for orthogonal U, V
nonneg diag Σ_1

Pf outline: largest e-value of M

① Let $\lambda_1 = \max \frac{x^T M x}{x^T x}$ } Rayleigh quotient

and v_1 - corresponding unit vector x

② Show that v_1 is an e-vector of eigenvalue λ_1

(pf: calculus - gradient $= 0$)

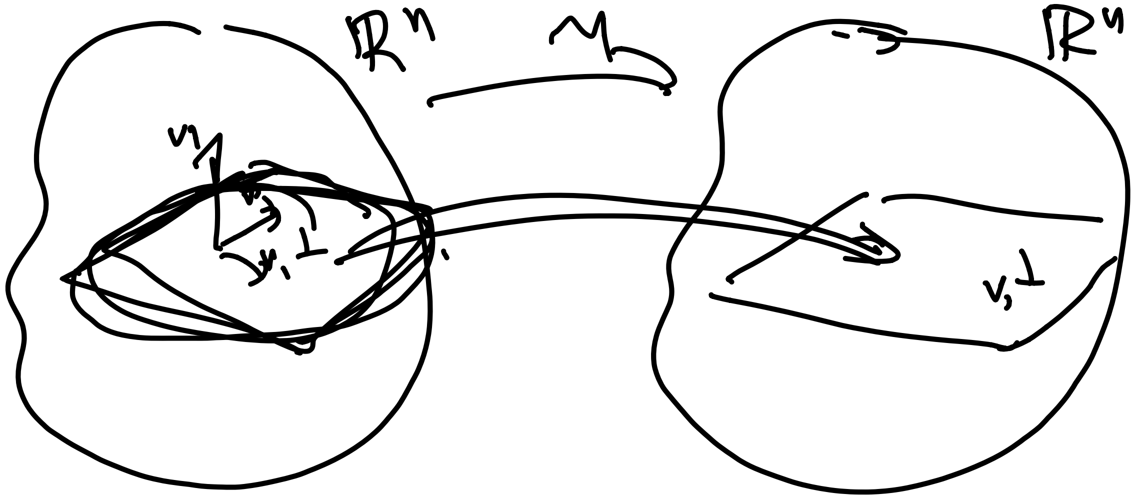
③ Restrict to $(n-1)$-dimensional subspace

$v_1^\perp = \{w : w^T v_1 = 0\}$

check: $M v_1^\perp \subseteq v_1^\perp$

$(v_1^T M w = \lambda_1 v_1^T w = 0)$

④ Apply induction on dimension to $M\big|_{v_1^\perp}$ = M restricted to $v_1^\perp$



- Get orthonormal basis of e-vectors on $v_1^\perp$
- Add v_1 to get basis of $\mathbb{R}^n$.

Computational Complexity of Linear Algebra

On $n \times n$ matrices all of the following can be computed in $\tilde{O}(n^\omega)$ arithmetic ops:

hide polylog(n) factors

$(2 \le \omega \le 2.373)$

- matrix mult.
- matrix inversion
- determinant
- characteristic polynomial
- eigendecomposition of symmetric matrices ? to within

- solving $Ax=b$ • SVD) arbitrary accuracy

For sparse matrices, prefer f(# nonzero entries)

Let G be undirected (for rest of today).
We know $\vec{1}$ is an evector of L of evalue 0.

<u>Q:</u> what can we say about the other evalues?

<u>A:</u> all nonnegative ($\Longleftrightarrow$ L pos. semi-definite)

$(0\,0 \cdots 0\,1\,0 \cdots 0)$

a

$$L = \sum_{(a,b)\in E} w_{a,b} L_{G_{(a,b)}} = \sum_{(a,b)\in E} w_{a,b} (\delta_a - \delta_b)(\delta_a-\delta_b)^T$$

graph w/ one edge

$$x^T L x = \sum_{(a,b)} w_{a,b} (x_a - x_b)^2 \geq 0$$

Let $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be evalues

<u>Thm:</u> $\lambda_2 = 0 \iff G$ disconnected

<u>Pf:</u> $\Leftarrow$ $L = \begin{pmatrix} L_{G_1} & 0 \\ 0 & L_{G_2} \end{pmatrix}$

e-values (L) = e-values$(L_{G_1}) \cup$ e-values(L_{G_2})

$\Rightarrow$ $\sum_{(a,b)\in E} w_{a,b} (x_a - x_b)^2 = 0$

$\Rightarrow$ $x_a = x_b$ for every edge

$\Rightarrow$ x is a mult. $\vec{1}$

G connected

$\Rightarrow$ only evectors of e-value 0 are multiple of $\vec{1} = v_1$

Intuition: larger $\lambda_2 \Rightarrow G$ more "highly connected".

Exercise: • calculate $\overset{=d=n}{\lambda_2}$ for the complete graph on n vertices w/ self-loops (degree n)

• is this the largest possible value of λ_A/d? ~~As → n a bipartite graph~~ No- in a bip. graph

for a d-regular graph

$$\frac{\lambda_n}{d} = 2$$

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & & & \vdots & & & \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} = 11^T$$

$$L = \begin{pmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & n-1 & -1 & \cdots & -1 \\ -1 & -1 & \ddots & & -1 \\ -1 & -1 & \cdots & n-1 & -1 \\ -1 & -1 & \cdots & -1 & n-1 \end{pmatrix}$$

$\mu_1 = n \quad \mu_2 = 0$

$$\lambda_2 = d - \mu_2 = d = n$$

$$L = D - M = dI - M$$

Q: characterization of λ_2 in terms of quadratic form $x^T L x$? or Rayleigh quotient $\frac{x^T L x}{x^T x}$?

$$\lambda_2 = \min_{x \perp \vec{1}} \frac{x^T L x}{x^T x} = \min_{\substack{S \subseteq \mathbb{R}^n \\ S \text{ 2-dim subspace}}} \max_{x \in S \setminus \{0\}} \frac{x^T L x}{x^T x}$$

$x^T L x = \sum w_{a,b} (x_a - x_b)^2$

Courant-Fischer Thm

Hall's Graph Drawing Method

find $f: V \to \mathbb{R}^2$ minimizing

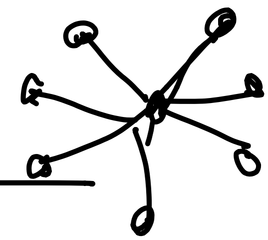
$$\sum_{(a,b) \in E} w_{a,b} \cdot \|f(a) - f(b)\|^2 = \sum_{(a,b) \in E} w_{a,b} \left\| \begin{pmatrix} x(a) \\ y(a) \end{pmatrix} - \begin{pmatrix} x(b) \\ y(b) \end{pmatrix} \right\|^2$$

s.t. $\|x\| = \|y\| = 1$, $x \perp \vec{1}$, $y \perp \vec{1}$, $x \perp y$

opt: $x = v_2$
$y = v_3$

Wilf's Theorem:

Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ be the eigenvalues of M. Then G can be properly colored with $\lfloor \mu_1 \rfloor + 1$ colors.



cf. – PSO : $d_{max}+1$ colors

– Fact : $\mu_1 \leq d_{max}$

– $\mu_1 = d_{max} \Leftrightarrow G$ regular

Proof of Wilf's Thm:

by induction on n

(1) find a vertex v of degree $\leq \lfloor \mu_1 \rfloor$

(2) color the rest of the graph w/ $\lfloor \mu_1 \rfloor + 1$ colors

(3) color v w/ any color not used by a neighbor

(1) Lemma: in any graph G,

$$d_{avg} \le \mu_1$$

Pf: $$\mu_1 = \max_x \frac{x^T M x}{x^T x} \ge \frac{\mathbb{1}^T M \mathbb{1}}{\mathbb{1}^T \mathbb{1}} = d_{avg}$$

(2) by induction hypothesis + Lemma: $G \setminus \{v\}$ has largest e-value $\le \mu_1$

Pf: $$\mu_1 = \max_{x \in \mathbb{R}^V} \frac{x^T M x}{x^T x}$$

$$\ge \max_{y \in \mathbb{R}^{V-\{v\}}} \frac{\begin{pmatrix} y \\ 0 \end{pmatrix}^T M \begin{pmatrix} y \\ 0 \end{pmatrix}}{\begin{pmatrix} y \\ 0 \end{pmatrix}^T \begin{pmatrix} y \\ 0 \end{pmatrix}}$$

$$= \max_{y \in \mathbb{R}^{V-\{v\}}} \frac{y^T M_{G-\{v\}}\, y}{y^T y} = \mu_1(G-\{v\})$$

vth component

Cauchy's Interlacing Thm:

A symmetric $n \times n$ matrix w/ e-values $\alpha_1 \geq \cdots \geq \alpha_n$

B principal $(n-1)\times(n-1)$ submatrix of –

w/ e-values $\beta_1 \geq \cdots \geq \beta_{n-1}$

Thm $\alpha_1 \geq \beta_1 \geq \alpha_2 \geq \beta_2 \geq \cdots \geq \beta_{n-1} \geq \alpha_n$