Announcements

- start recording
- scribe: work w/other scribe to produce one set of notes
- My OH: Mon 12:30-1:30, Thu 9-10
- TF hybrid section/OH: Mon 2-3, Wed 2-3, Wed 5-6, Thu 5:30-6:30
- PS 2 posted, including final project ideas problem
- If Zoom goes down, check Piazza
- sync whiteboard
- gather after class: link in chat
- jamboard link in chat

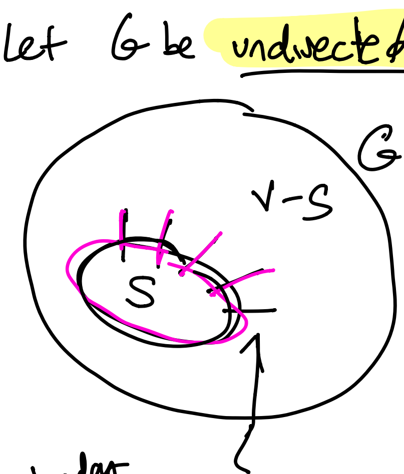
Agenda

- Recap: conductance
- Examples
- Cheeger's Inequality and proof
- if time: higher-order Cheeger

Isoperimetry + Conductance

Let G be undirected

How well is S connected to $V-S$? for $|S| \le 1/2$



Algorithmic motivation:

"Sparsest Cut" problem: NP-hard
given G, find S minimizing $\phi(S)$

boundary
$$\partial S \overset{\text{def}}{=} \{(a,b) \in E : a \in S, b \notin S\}$$

$\rightarrow$ partition G cutting few edges $\Rightarrow$ divide + conquer

---

Conductance
$$\phi(S) = \frac{|w(\partial S)|}{d(S)}$$

$$w(\partial S) = \sum_{e \in \partial S} w(e)$$

$$d(S) = \sum_{a \in S} d(a)$$

$$= \Pr\left[\begin{array}{l}\text{r.w. started at } \pi|_S \\ \text{leaves } S \text{ in one step}\end{array}\right]$$

$\pi|_S$ = stationary conditional on S

makes sense for digraphs

$$\in \frac{\Pr\left[\begin{array}{l}\text{r.w. started at } \pi \\ \text{has 1st vertex in } S \text{ and 2nd vertex in } V-S\end{array}\right]}{\pi(S)\cdot\pi(V-S)} \cdot [1,2]$$

when $\pi(S) \le 1/2$

$$\| \quad \tilde{\phi}(S)$$

$$\pi(S) = \sum_{a \in S} \pi(a)$$

$$\phi(G) \overset{\text{def}}{=} \min_{S:\, d(S) \le d(V)/2} \phi(S)$$ conductance of G

dumbbell   cycle

Thm: $$\frac{\nu_2}{2} \le \phi(G) \le \sqrt{2\nu_2}$$

last time   Cheeger's Inequality

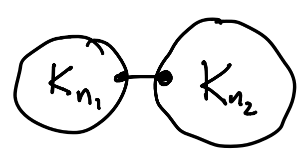
cf. $$\frac{\log\left(\frac{1}{\varepsilon}\right)}{\nu_2} \le \tilde{t}_{mix}(\varepsilon) \le \frac{O\left(\log\left(n d_{max}/d_{min}\, \varepsilon\right)\right)}{\nu_2}$$

ex. on ps2   lazy r.w.

Exercise: estimate (up to log factors) $\phi(G)$ and ν_2 (possibly via mixing time)

1) n-cycle

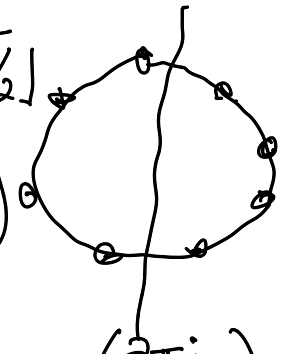
2) imbalanced dumbbell



$n_2 \ge n_1$

1) n-cycle $\quad \phi(C_n) = \frac{2}{2 \cdot \lfloor n/2 \rfloor}$

$= \Theta\left(\frac{1}{n}\right)$



$\nu_2(C_n) = 1 - \omega_2 = 1 - \cos\left(\frac{2\pi i}{n}\right)$

$= \Theta\left(\frac{1}{n^2}\right)$

mixing time $= \Theta(n^2)$

std deviation of a ± 1 r.w. of length t is $O(\sqrt{t})$

---

2) dumbbell

$w(\partial S) = 1$

$d(S) = (n_1 - 1)^2 + n_1 \quad \approx n_1^2$

$\phi(G) \doteq \phi(S) \doteq \Theta\left(\frac{1}{n_1^2}\right)$

stating prob on $S \approx \frac{n_1^2}{n_1^2 + n_2^2}$

mixing time $= \Theta(n^2)$

$\nu_2 = \tilde{\Theta}\left(\frac{1}{n_1^2}\right)$

" on $V-S \approx \frac{n_2^2}{n_1^2 + n_2^2}$

Proof of "easy" direction: $\phi(G) \geq \nu_2/2$

$$\tilde{\phi}(S) = \frac{w(\delta S)/d(V)}{\left(\frac{d(S)}{d(V)}\right)\cdot\left(\frac{d(S-V)}{d(V)}\right)}$$

$$= \frac{y_S^T L y_S}{y_S^T D y} \quad \text{where } y_S = \left(\vec{1}_S - \frac{d(S)}{d(V)}\cdot\vec{1}\right) \perp \vec{d}$$

$$= \frac{z_S^T N z_S}{z_S^T z_S} \quad \text{where } z_S = D^{1/2} y_S \perp \vec{d}^{1/2}$$

$$\left.\begin{aligned} \phi(G) &= \min_{d(S)\leq d(V)/2} \phi(S) \\ &\geq \frac{1}{2}\min_S \tilde{\phi}(S) \\ &= \frac{1}{2}\min_S \frac{z_S^T N z_S}{z_S^T z_S} \end{aligned}\right\} \text{NP-hard combinatorial optimization}$$

$$\left.\begin{aligned} &\geq \frac{1}{2}\min_{\substack{\vec{z}\perp \vec{d}^{1/2} \\ \vec{z}\neq 0}} \frac{z^T N z}{z^T z} \\ &= \frac{1}{2}\nu_2 \end{aligned}\right\} \text{poly-time continuous optimization}$$

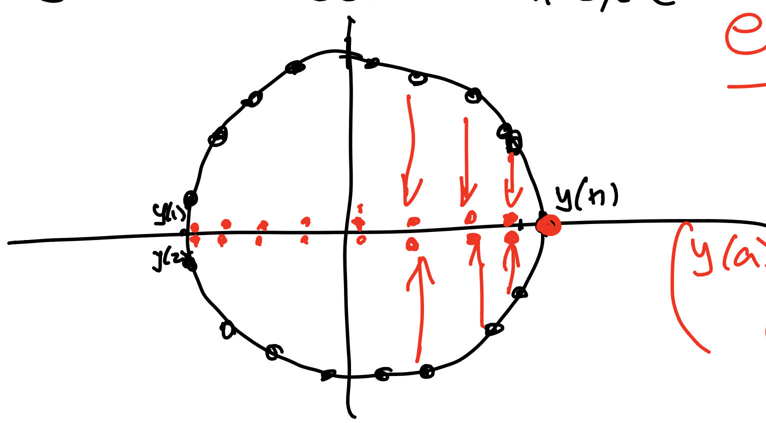
Proof of Cheeger's Inequality

Goal: Given $\vec{y} \perp \vec{d}$ s.t. $\frac{y^T L y}{y^T D y} \le \rho$ (e.g. $\vec{y} = D^{1/2}\psi_2$, $\rho = \nu_2$)

"round" $\vec{y}$ to obtain a set S (i.e. a $\{0,1\}$ vector 1_S)

s.t. $\frac{w(\partial S)}{\min\{d(S), d(V-S)\}} \le \sqrt{2\rho}$.

Example: $\vec{y}$ = real 2nd evector of n-cycle



$$\frac{e^{-2\pi i a/n} + e^{+2\pi i a/n}}{2}$$

$$\| $$

$y(a) = \cos\left(\frac{2\pi i a}{n}\right)$ before sorting.

sort $y(1) \le y(2) \le \cdots \le y(n)$

will ~~take~~ show: can take a "threshold cuts"

$$S_\tau = \{a : y(a) \le \tau\}$$

"Fiedler's Alg": try all prefixes of sorted coordinates

In fact, ~~will define a distribution on thresholds~~ t

s.t. $\underline{\mathbb{E}_{\tau}\left[w(\partial S_\tau)\right]} \le \underline{\sqrt{2\rho}} \cdot \underline{\mathbb{E}_{\tau}\left[\min\{d(S_\tau), d(V-S_\tau)\}\right]}$

Step 1: center $\vec{y}$

- $\vec{z} = \underline{\vec{y} - s\vec{1}}$ for appropriate s

  so that $\sum_{a:z(a)<0} d(a) \le \frac{d(V)}{2}$ and $\sum_{a:z(a)>0} d(a) \le \frac{d(V)}{2}$

- $$\frac{\vec{z}^T L \vec{z}}{\vec{z}^T D \vec{z}} = \frac{y^T L \vec{y} - \cancel{s\vec{y}^T L \vec{1}} - \cancel{s\vec{1}^T L \vec{y}} + \cancel{s^2 \vec{1}^T L \vec{1}}}{y^T D \vec{y} - \cancel{s\vec{y}^T D \vec{1}} - \cancel{s\vec{1}^T D \vec{y}} + \underbrace{s^2 \vec{1}^T D \vec{1}}_{\ge 0}}$$

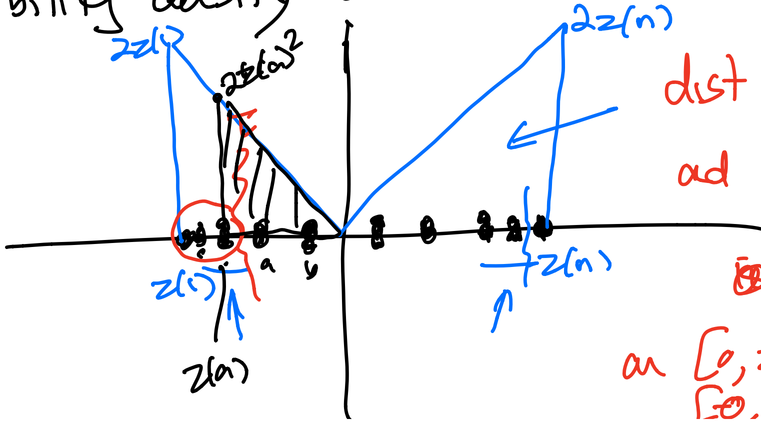
  $\vec{y} \perp \vec{d}$

  $$\le \frac{y^T L y}{y^T D y} \le \rho$$

- Assume WLOG $z(1)^2 + z(n)^2 = 1$

Step 2: distribution on τ

- Probability density at $t = 2|t|$



dist of $t^2 \mid t>0$

and dist of $t^2 \mid t<0$

~~see~~ uniform

on $[0, z(n)^2]$ and $[0, z(1)^2]$

Claim: $\mathbb{E}_{\tau}\left[\min\{d(S_\tau), d(V-S_\tau)\}\right] = z^T D z$

Pf: $\text{LHS} = \sum_a d(a) \cdot \Pr[a \text{ is in "smaller" of } S_\tau \text{ and } V-S_\tau]$

centering

$$= \sum_a d(a) \cdot \Pr[\tau \text{ is between } z(a) \text{ and } 0]$$

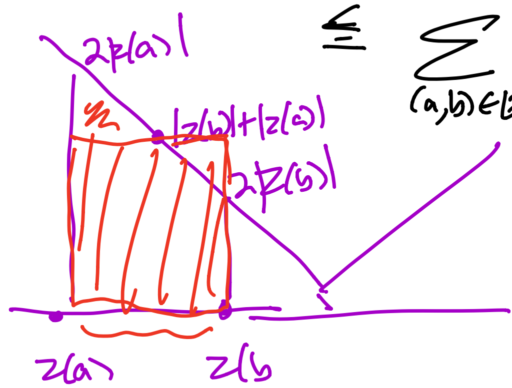
$$= \sum_a d(a) \cdot z(a)^2$$

$$= z^T D z$$

Claim: $\mathbb{E}_{\tau}\left[w(\partial S_\tau)\right] \le \sqrt{2}\sqrt{z^T L z} \cdot \sqrt{z^T D z}$

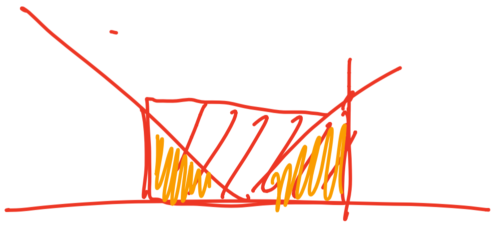
Pf: $\text{LHS} = \sum_{(a,b)\in E} w_{ab} \cdot \Pr[\tau \text{ is between } z(a) \text{ and } z(b)]$

$$\le \sum_{(a,b)\in E} w_{ab} \cdot |z(b)-z(a)| \cdot (|z(b)|+|z(a)|)$$



$$\le \ldots$$

Cauchy-Schwartz



Higher-order Cheeger:

Fact: $\lambda_k = 0 \iff G$ has at least k connected components

Def: $\phi_k(S_1, \dots, S_k) = \max_{j=1,\dots,k} \phi(S_j)$

$$\phi_k(G) = \min_{\substack{S_1,\dots,S_k \\ \text{disjoint}}} \dots$$

Thm: $\frac{\lambda_k}{2} \le \phi_k(G) \le \mathrm{poly}(k) \cdot \sqrt{\lambda_k}$