Announcements

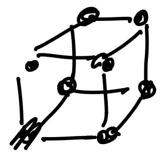
- start recording
- scribe: work w/other scribe to produce one set of notes
- My OH: Mon 12:30–1:30, Thu 9–10
- TF hybrid section/OH: Mon 2–3, Wed 2–3, Wed 5–6, Thu 5:30–6:30
- PS 1 posted, due Fri 9/25
- If Zoom goes down, check Piazza
- Sync whiteboard

Agenda

1) Corrections + Recap
2) Mixing of Random Walks on Undirected Graphs
3) Random Walks on Directed Graphs
4) Markov Chain Monte Carlo

Correction: eigenvalues of undirected n-cycle adjacency matrix $= \omega^{-r} + \omega^{r} = 2\cos\left(\frac{2\pi r}{n}\right)$, $r = 0, \ldots, n-1$

Example 1: Boolean hypercube $\quad G = \underbrace{\mathbb{Z}_2^d}_{\substack{\{0,1\}^d \\ \text{w/ addition mod } 2}} \quad S = \{e_1, e_2, \dots, e_d\}$ (i.e. $W = 1_S$)



Fourier basis: for each $r \in \mathbb{Z}_2^d$

$$\chi_r(x) = (-1)^{\langle r, x\rangle} = (-1)^{\langle r, x\rangle \bmod 2}$$

$$\hat{W}_r = \sum_{s \in \mathbb{Z}_2^d} 1_S(s) \cdot (-1)^{\langle r, s\rangle}$$

$$= \sum_{i=1}^{d} (-1)^{\langle r, e_i\rangle} = \sum_{i=1}^{d} (-1)^{r_i}$$

$$= \#(0\text{'s in } r) - \#(1\text{'s in } r) = d - 2\underbrace{|r|}_{\text{Hamming weight}}$$

$L = D - M = dI - M$

$\lvert r\rvert$	# e-values	e-value of M	e-value of L	e-value of W	e-value of N
0	1	d	0	1	0
1	d	$d-2$	2	$1-2/d$	$2/d$
2	$\binom{d}{2}$	$d-4$	4	$1-4/d$	$4/d$
3	$\binom{d}{3}$	$d-6$	6	$1-6/d$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
		2		$2/d$	
$d/2$	$\binom{d}{d/2}$	0	d	0	$\vdots$
$\vdots$	$\vdots$	-2	$d+2$	$-2/d$	$\vdots$
		$\vdots$	$\vdots$	$\vdots$	
$d-1$	d	$-(d-2)$	$\vdots$	$-1+2/d$	$\vdots$
d	1	$-d$	$2d$	-1	2

Perron-Frobenius Thm for Symmetric Matrices

Let M = symmetric, nonnegative real matrix (e.g. adjacency or normalized adj mx of undirected G)

w/corresponding graph G (i.e. $E = \{(a,b) : M(a,b) > 0\}$.

Eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$

Then 1) $\exists$ ~~strictly positive~~ nonnegative e-vector v_1 w/ e-value μ_1

~~(and the only nonneg. e-vectors are multiples of v_1)~~ + strict positivity of v_1

2) $\mu_1 \geq \mu_2$ with strict inequality if G connected

3) $\mu_1 \geq -\mu_n$ with strict inequality if G connected + nonbipartite

Moreover, if G bipartite then $\mu_{n-i} = -\mu_{i+1}$ for $i = 0, \ldots, n-1$

Random Walks on Undirected Graphs

$M(a,b) = w(b,a)$.

$n = \#$ vertices

G undirected, weighted. $W = MD^{-1}$

$p_0 =$ initial probability distribution on vertices $\in \mathbb{R}^n$

$p_t =$ prob dist. after t steps of r.w.

$= W^t p_0$

How to analyze?

<u>Claim</u>: W has a basis of e-vectors $\phi_1, \ldots, \phi_n$
w/ e-values $\omega_1 \geq \omega_2 \geq \omega_3 \geq \cdots \geq \omega_n$

<u>Pf</u>: $W = D^{1/2} A D^{-1/2}$ for normalized adj mx $A = D^{-1/2} M D^{-1/2}$
which is symmetric

$\Rightarrow A$ has orthonormal basis of e-vectors $\psi_1, \ldots, \psi_n$
with e-values $\omega_1 \geq \omega_2 \geq \omega_3 \geq \cdots \geq \omega_n$

$\Rightarrow \phi_i = D^{1/2} \psi_i$ is basis of e-vectors for W $\square$

(not necessarily orthogonal when G irregular)

Write $p_0 = c_1\phi_1 + c_2\phi_2 + \cdots + c_n\phi_n$

$$W^t p_0 = \underbrace{c_1 \omega_1^t \phi_1}_{\Pi} + c_2 \omega_2^t \phi_2 + \cdots + c_n \omega_n^t \phi_n$$

Will be dominated by eigenvalue(s) of largest magnitude

$\omega_1 = 1$ because $\psi_1 = \vec{d}^{1/2} / \|\vec{d}^{1/2}\|$ is strictly pos e-vector of A of e-value 1. By P-F (on each component) $|\omega_i| \leq 1$ for $i = 2, \ldots, n$.

$$\phi_1 = D^{1/2}\psi_1 = \frac{\vec{d}}{\|\vec{d}^{1/2}\|}$$

$$c_1 = \underbrace{(D^{-1/2}p_0)}^T\psi_1 = p_0^T D^{-1/2}\frac{\vec{d}^{1/2}}{\|\vec{d}^{1/2}\|} = \frac{p_0^T\vec{1}}{\|\vec{d}^{1/2}\|} = \frac{1}{\|\vec{d}^{1/2}\|}$$

$$c_1\phi_1 = \frac{\vec{d}}{\|\vec{d}^{1/2}\|^2} = \frac{\vec{d}}{\sum_a d(a)} \overset{\text{def}}{=} \pi \quad \text{stationary dist.} \quad (W\pi = \pi)$$

Thm: $$\max_{j>1}|\omega_j| = \max_{\substack{\text{prob. dist}\\ p}} \frac{\|Wp-\pi\|_{\pi^{-1}}}{\|p-\pi\|_{\pi^{-1}}} \overset{\text{def}}{=} \omega_\pi$$

where $$\|x\|_{\pi^{-1}} \overset{\text{def}}{=} \sqrt{\sum_{a=1}^{n}\frac{x(a)^2}{\pi(a)}}$$

Proof: $$\max_{j>1}|\omega_i| \overset{\text{ps1}}{=} \max_{x\perp\psi_1}\frac{\|Ax\|}{\|x\|} \qquad (A = D^{-1/2}WD^{1/2})$$

$$= \max_{y:\,D^{-1/2}y\perp\psi_1}\frac{\|D^{-1/2}Wy\|}{\|D^{-1/2}y\|} \qquad (x = D^{-1/2}y)$$

$$= \max_{y:\,\sum_a y_a = 0}\frac{\|Wy\|_{\vec{d}^{-1}}}{\|y\|_{\vec{d}^{-1}}} \qquad p = \pi + cy$$

$$= \max_{\text{prob dist } p}\frac{\|W(p-\pi)\|_{\vec{d}^{-1}}}{\|p-\pi\|_{\vec{d}^{-1}}}$$

$$= \max_{\text{prob dist } p}\frac{\|Wp-\pi\|_{\pi^{-1}}}{\|p-\pi\|_{\pi^{-1}}}$$

By def: $\| W^t p_0 - \pi \|_{\pi^{-1}} \le \omega_\pi^t \cdot \| p_0 - \pi \|_{\pi^{-1}}$

- initial distance $\| p_0 - \pi \|_{\pi^{-1}} \le \| p_0 \|_{\pi^{-1}} \le \sqrt{\dfrac{1}{\pi_{min}}}$

- final ℓ_1 distance ($= 2 \cdot$ total variation distance)

$$|W^t p_0 - \pi|_1 = \sum_a \frac{|W^t p_0(a) - \pi(a)|}{\pi(a)^{1/2}} \cdot \pi(a)^{1/2}$$

$$\underset{\text{Cauchy-Schwarz}}{\le} \left(\sum_a \frac{(W^t p_0(a) - \pi(a))^2}{\pi(a)} \right)^{1/2} \cdot 1 = \| W^t p_0 - \pi \|_{\pi^{-1}}$$

Thm: for every start distribution p_0

$$|W^t p_0 - \pi|_1 \le \omega_\pi^t \cdot \| p_0 \|_{\pi^{-1}} \le \omega_\pi^t \cdot \frac{1}{\sqrt{\pi_{min}}}$$

where $\pi_{min} = \dfrac{d_{min}}{\sum_a d(a)} \in \left[\dfrac{d_{min}}{d_{max}\, n}, \dfrac{1}{n} \right]$

Cor: "Mixing time" to get to within ℓ_1 dist. ε

is $t = O\left(\dfrac{\log \left(n d_{max} / \varepsilon d_{min} \right)}{1 - \omega_\pi} \right)$

Q: how to ensure $\omega_\pi < 1$?

A: $\omega_\pi = \max\{\omega_2, -\omega_n\}$

$\omega_2 < 1$ iff G connected

$\omega_n > -1$ iff G nonbipartite

Lazy Random Walk : $\tilde{W} = \frac{1}{2}W + \frac{1}{2}I$

$\tilde{\omega}_j = \frac{1+\omega_j}{2} \geq 0$, so $\tilde{\omega}_\pi = \tilde{\omega}_2 = \frac{1+\omega_2}{2}$

and $1-\tilde{\omega}_\pi = \frac{1-\omega_2}{2} = \frac{\nu_2}{2}$

$\nu_2 = 2^{nd}$ smallest e-value of normalized laplacian $N = I - A$

Exercise: bound mixing time of lazy + non-lazy random walks on hypercube combinatorially and using e-values, and compare.

Let k = dimension of hypercube

$\omega_1 = 1 \quad \omega_2 = 1 - 2/k \quad \omega_n = -1 \qquad \pi$ = uniform on 2^k vertices

Non-lazy $\quad \omega_\pi = 1 \Rightarrow$ no mixing

Lazy: $\quad \tilde{\omega}_\pi = \frac{1+\omega_2}{2} = 1 - 1/k \qquad \varepsilon$ const.

$$t_{mix} = O\left(\frac{\log(n/\varepsilon)}{1-\tilde{\omega}_\pi}\right) = O(k^2)$$

Combinatorially $t_{mix} = O(k \log k)$ (time to select all coordinates)

<u>Converse</u> (maybe ps 2): $\forall$ undirected G $\exists$ start vertex s such that for $p_0 = \delta_s$:

$$t = \Omega\left(\frac{\omega_\pi}{1-\omega_\pi} \cdot \log\left(\frac{1}{\varepsilon}\right)\right)$$

steps are needed to get to within ℓ_1 distance ε of π.

Also: for every undirected, connected, nonbipartite G,

$\omega_\pi \leq 1 - 1/\text{poly}(n \cdot d_{max})$ (cf. ps 1)

so mixing time $\leq \text{poly}(n \cdot d_{max})$.

Random Walks on Directed Graphs

<u>Q</u>: what changes?

<u>Q</u>: what happens in non-connected digraphs?