Announcements

- start recording
- scribe: work w/team, scribe to produce one set of notes
  - My CBT: Mon 12:30-1:30, Thu 9-10 (Mon. holiday)
  - TG hybrid sections: Mon 2-3, Wed 2-3, Wed 5-6, Thu 5:30-6:30
- PS 2 due Fri, including final project ideas problem
- If Zoom goes down, check Piazza
- Sync whiteboard
- Among Us Fri night
- No jamboard today
- Last chance to fill out feedback - Q on Recsys & readings was supposed to be "how useful"

Agenda

- Power Method (finish)
- Expander Graphs: Measures of Expansion
Fiedler's Algorithm for Sparsest Cut

Given weighted, undirected $G_1$, find $S \subseteq V$ \( w/d(S) \leq d/\sqrt{\lambda_2} \)
and \( \phi(S) \leq \sqrt{\lambda_2} = O(\sqrt{\lambda_2}) \)

1. Find a vector $y \in \mathbb{R}^n$ such that \( y^T D y \leq 2 \) by eigendecomposition of $N$

2. Sort $y_1 \leq y_2 \leq ... \leq y_n$ \( O(n \log n) \)

3. Find $k \in \{1, ..., n\}$ minimizing

\[
\frac{w(e_1, ..., e_k, e_{k+1}, ..., e_{d/\sqrt{\lambda_2}})}{\min \{d(e_1, ..., e_k), d(e_{k+1}, ..., e_{d/\sqrt{\lambda_2}})\}}
\]

\( O(m) \)

where $e(S,T) = \sum_{(a,b) \in S \times T} \alpha S \backslash \beta T$
How to speed up Step 1? Goal: time $O(n+m)$

**Power Method**

1. **Estimate**
   - Largest eigenvalue of a PSD matrix $M$

   \[
   \begin{align*}
   &\text{1. Choose } x \in \mathbb{R}^n \text{ with } x \perp \text{eigen vectors of } M \\
   &\text{2. Let } y = M^k x \text{ for } k = O(\frac{\log(n/\epsilon)}{\epsilon}) \text{ time} \\
   \end{align*}
   \]

   $O(k \cdot m)$

   \[
   \text{Then: with constant probability, } M_1 \geq \frac{y^T M y}{y^T y} \geq (1-\epsilon)^3 \cdot M_1
   \]
II. 2nd largest eigenvalue of a psd matrix

0. Let \( x \) be \( \mathcal{E} \pm I \)

1. \( x_0 = x - \langle x, v_1 \rangle v_1 \)

3. \( y_k = M^k x_0 \quad \text{for} \quad k = O\left(\frac{\log n}{\varepsilon}\right) \)

Cor: w/ constant probability

\[
\mu_2 \geq \frac{y^T M y}{y^T y} \geq \mu_2 \cdot (1 - \varepsilon)
\]

and \( y \perp v_1 \)
III. Second Smallest E-value of Normalized Laplacian

Apply II to matrix

\[ 2 I - N = I + A = 2 D^{-\frac{1}{2}} \tilde{W} D^\frac{1}{2} \text{ psd} \]

\[ \text{Eigenvectors: } 2, 1 + \omega_2, 1 + \omega_3, \ldots, 1 + \omega_n \]
\[ 2 - \omega_2, 2 - \omega_3, \ldots \]

Obtain vector \( y \perp d \frac{\sqrt{2}}{} \) s.t.

\[ 2 - \frac{y^T N y}{y^T y} = \frac{y^T (2 I - N) y}{y^T y} \geq (1 - \varepsilon)(2 - \omega_2) \]

i.e.

\[ \frac{y^T N y}{y^T y} \leq (1 + \varepsilon) \omega_2 + 2 \varepsilon \]

Take \( \varepsilon = \frac{\omega_2}{4} \)

\[ \Rightarrow \text{time } O (m \cdot \frac{\log(n \omega_2)}{\omega_2}) \]
Better: apply power meth to
"pseudoinverse" $N^+$

eigenvalues $0, \frac{1}{\nu_2}, \frac{1}{\nu_3}, \ldots, \frac{1}{\nu_n}$

\[
(1-\epsilon) \frac{1}{\nu_2} \leq \frac{1}{(1+\epsilon)\nu_2}
\]

Later in course: algorithms that gain $G$ and
and $x$, compute $N^+x$ in time $O(m+n)$

$\Rightarrow$ find cut of conductance $O(\sqrt{\nu_2})$
in time $O(m+n)$

nearly linear time
Expander Graphs Many applications in TCS!
d-regular n-vertex digraphs, $n \to \infty$

**Sparse:** typically $d = O(1)$
(sometimes $d = \text{polylog}(n)$)

**Well-connected:** several defs

1) spectral expansion $\gamma^* = \gamma(\omega) = 1 - \omega(\omega) \geq \gamma$

\[
\gamma(\omega) = \max_p \frac{\|W_p - u\|}{\|p - u\|} = \max_{x \in u} \frac{\|Wx\|}{\|x\|} = \max_{\omega, \omega^*} \frac{\|\omega - \omega^*\|}{\omega^*} \geq \omega_2
\]

```
\omega(\omega) = 1
\gamma(\omega) > 0
\omega(\omega) = \Theta(1/n) Ideally maximize $\gamma = 1 - \omega$
```

Typically want $\gamma > 0$ independent of $n$. 

```
\omega(\omega) = \Theta(1/n)
```

Complete graph as a function of $d$

$\delta(\omega) = 1 - \frac{1}{n-1} \sqrt{\det(-L)}$
2) \((k, a)\) vertex expansion

\[ A \subseteq S \subseteq V, \quad |N(S)| \geq a \cdot |S| \]

Typically \(k = \Omega(n), \quad a = 1 + \Omega(1)\)

Ideally maximize \(a\) as function of \(n, k, d\).

\[ a \geq 1 \quad \text{by regularity} \]

es.

\[ a = d - 1.01 \]

3) \((k, \varepsilon)\) edge expansion

\[ A \subseteq S \subseteq V, \quad |e(S, S^c)| \geq \varepsilon \cdot d \cdot |S| \]

Typically \(k = \Omega(n), \quad \varepsilon = \Omega(1)\)

Ideally maximize \(\varepsilon\) as function of \(n, k, d\)

\[ \phi(S) = \frac{|e(S, S^c)|}{d \cdot |S|} \]
Theorem: Let $\mathcal{G}$ be an infinite family of regular, undirected graphs.

The following are equivalent:

1) \( \exists \, \varepsilon > 0 \) s.t. every \( G \in \mathcal{G} \) has spectral expansion at least \( \varepsilon \).

2) \( \exists \, \varepsilon > 0 \) s.t. every \( G \in \mathcal{G} \) is an \((N/2, \varepsilon)\) edge expander.

Proof: By Cheeger’s Inequalities

\[
\frac{\chi(G)}{2} \leq \frac{\lambda_2}{2} \leq \phi(G) \leq \sqrt{2\lambda_2} \leq \sqrt{2\chi(G)}
\]

\[
\min \phi(G) \text{ if } G \text{ lazy}
\]

\( \chi(G) \) def \( 1 - \omega(G) = 1 - \max \{ \sum w_z, -\operatorname{cut} \} \)

\[
= \min \{ \frac{1}{2} \sum w_z, 1 + \operatorname{cut} \}
\]

\[
= \frac{1}{2} \sum w_z, 1 + \operatorname{cut}
\]

\( \min \frac{1}{2} \sum w_z, 1 + \operatorname{cut} \)

\( = 1/2 \) (\( \varepsilon \geq 0 \) if \( G \) lazy)
Vertex Expansion vs. Spectral Expansion

**Thm:** Let \( \mathcal{G} \) be an infinite family of \( d \)-regular, undirected graphs.

The following are equivalent "Expandable Graphs"

1) \( \exists \, \varepsilon > 0 \) s.t. every \( G \in \mathcal{G} \) has spectral expansion \\
   at least \( \varepsilon \).

2) \( \exists \, \varepsilon > 0 \) s.t. every \( G \in \mathcal{G} \) is a \( (n/2, \varepsilon) \)-vertex expander.

bipartite graphs are not vertex expanders

\[ \gamma(G) \overset{\text{def}}{=} |1 - \omega(G)| = 1 - \max \left\{ \frac{\varepsilon}{d} w_2, -cn^2 \right\} \]
\[ = \min \left\{ \frac{\varepsilon}{d} \left(1 - w_2\right), 1 + cn^2 \right\} \]
\[ = \min \left\{ \frac{\varepsilon}{d} \frac{\varepsilon}{2}, 1 + cn^2 \right\} \]
\[ = \frac{\varepsilon}{2} \quad (c_n \geq 0 \text{ if } G \text{ lazy}) \]