Announcements

- start recording
- scribe: work w/other scribe to produce one set of notes
- My OH: Mon 12:30-1:30, Thu 9-10
- TF section/OH: Mon 2-3, Tue 8-9, Wed 5-6, Thu 5:30-6:30
- PS 4 posted. Project proposal due Sun 11/8, problems due TUE 11/17
- If Zoom goes down, check Piazza
- Sync whiteboard

Agenda

- Matrix-Tree Theorem
- Leverage Scores

Matrix-Tree Theorem

Let G be a connected, weighted, undirected graph with Laplacian L.

Then $n \cdot \sum_{\substack{\text{spanning} \\ \text{trees } T \text{ of} \\ G}} w(T) = \sigma_{n-1}(L) \overset{\text{def}}{=} (-1)^{n-1} \cdot (\text{coeff of } x \text{ in } \det(xI - L))$

$= \sum_{i=1}^{n} \prod_{j \neq i} \lambda_j$

$= \boxed{\sum_{\substack{S \subseteq V \\ |S| = n-1}} \det(L(S,S))}$

$G=(V,E)$

$|T| = n-1$

$T \subseteq E$ s.t. $G=(V,T)$ is a (connected) tree

$\prod_{e \in T} w_e$

2017 (Schild): algs for sampling in time $m^{1+o(1)}$

2018 (Anari et al.): approx counting + sampling spanning forests

Corollaries:

- poly-time algorithm for counting spanning trees
- poly-time alg for sampling random spanning trees (sample T w.p. prop to $w(T)$)

→ Toss a biased coin to include edge e w.p. $\dfrac{\#\text{spanning trees in } G'}{\#\text{spanning trees in } G}$

- → if don't include e → remove e to get G' → recurse on G' to get rest
- → if include e → contract e to a single vertex to get G'' → recurse ~~on~~ e

Proof: Write $L = \sum_{(a,b) \in E} w_{ab} (\delta_a - \delta_b)(\delta_a - \delta_b)^T = U^T W U$

$= B^T B$ where $B = \left.\begin{pmatrix} \text{—} \\ \text{—} \\ \sqrt{w_{ab}}(\delta_a - \delta_b) \\ \vdots \\ \text{—} \end{pmatrix}\right\} m$, $\overbrace{}^{n}$, e, s

$\sigma_{n-1}(L) = \sigma_{n-1}(B^T B)$

$= \sigma_{n-1}(B B^T)$

$= \sum_{\substack{S \subseteq E \\ |S| = n-1}} \det((BB^T)(S,S))$

e-values of AB = e-values of BA

$$= \sum_{\substack{S\subseteq E\\ |S|=n-1}} \sigma_{n-1}\left((BB^T)(S,S)\right)$$

n-1 x n-1 matrix

$$= \sum_{\substack{S\subseteq E\\ |S|=n-1}} \sigma_{n-1}\left(B(S;\cdot)^T B(S;\cdot)\right)$$

$$= \sum_{\substack{S\subseteq E\\ |S|=n-1}} \sigma_{n-1}\left(L_{G_S}\right)$$

If S not a tree then G_S is disconnected $\Rightarrow$ 2 zero e-values $\Rightarrow \sigma_{n-1}(L_{G_S})=0$

$$= \sum_{\substack{T\subseteq E\\ T \text{ a spanning tree}}} \underline{\sigma_{n-1}\left(L_{G_T}\right)}$$

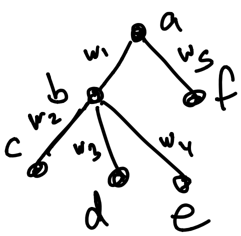
$$= \sum_{\substack{T\subseteq E\\ T \text{ a spanning tree}}} \underline{\sum_{a\in V}} \underline{\det\left(L_{G_T}(V-\{a\}, V-\{a\})\right)}$$

$$= \sum_T \sum_a \det\left(B(T,V-\{a\})^T B(T,V-\{a\})\right)$$

$$= \sum_T \sum_a \det\left(B(T,V-\{a\})\right)^2$$

$$= \sum_T \sum_a \left(\prod_{e\in T} \sqrt{w_e}\right)^2$$

$$B(T;V-\{a\}) = \begin{array}{c} \\ \\ \\ \\ \\ \\ a \end{array} \overbrace{\begin{array}{c} \cancel{\sqrt{w_1}} \\ \\ \\ \\ \\ \\ \end{array} \begin{pmatrix} -\sqrt{w_1} & & & & 0 \\ \sqrt{w_2} & -\sqrt{w_2} & & & \\ \sqrt{w_3} & & -\sqrt{w_3} & & \\ \sqrt{w_4} & & & -\sqrt{w_4} & \\ & & & & -\sqrt{w_5} \\ & & & & \ddots \\ b & c & d & e & f \end{pmatrix}}^{V-\{a\}} \begin{array}{l} e_1 \\ e_2 \\ e_3 \\ T \\ \\ \\ \\ \end{array}$$



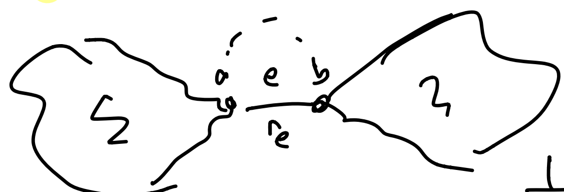
Recap: Effective Resistance

G undirected, weighted, connected graph.

View each edge as a resistor w/ resistance $r_e = \frac{1}{w_e}$

$$R_{eff}(a,b) = \left[v(a) - v(b) \text{ when } i_{ext} = \delta_a - \delta_b \right]$$

$$= (\delta_a - \delta_b)^T L^+ (\delta_a - \delta_b)$$



If removing e disconnects the graph, then

$\ell_e = 1$

$\frac{1}{r_e} = w_e = \frac{1}{R_{eff}(e)}$

Def: leverage score of edge e: $\ell_e = w_e R_{eff}(e)$

Thm: $\Pr[e \in T] = \ell_e$

T a random spanning tree chosen w.p. $\propto w(T)$

Proof: $\Pr_T[e \in T] = \frac{\sum_{T: e \in T} w(T)}{\sum_T w(T)}$

$$\overset{\text{matrix-Tree thm}}{=} \sum_{T: e \in T} \frac{\sigma_{n-1}(L_T)}{\sigma_{n-1}(L)}$$

$$= \sum_{\substack{S \subseteq E \\ |S|=n-1 \\ e \in S}} \frac{\sigma_{n-1}(L_S)}{\sigma_{n-1}(L)}$$

$$= \sum_{\substack{S \subseteq E \\ |S|=n-1 \\ e \in S}} \sigma_{n-1}(L_S L^+)$$

$$= \sum_{\substack{S \subseteq E \\ |S|=n-1 \\ e \in S}} \sigma_{n-1}(B(S;\cdot)^T B(S;\cdot) L^+)$$

$$= \sum_{\substack{S \subseteq E \\ |S|=n-1 \\ e \in S}} \sigma_{n-1}(B(S;\cdot) L^+ B(S;\cdot)^T)$$

$$= \sum_{\substack{S \subseteq E \\ |S|=n-1 \\ e \in S}} \sigma_{n-1}(\Gamma(S;S)) \quad m \times m$$

for $\Gamma = B L^+ B^T$

$\Gamma^2 = \Gamma$

$$= \sum_{\substack{S \subseteq E \\ |S|=n-1 \\ e \in S}} \underbrace{\sigma_{n-1}(\Gamma(S;\cdot)\Gamma(\cdot;S))}_{\det}$$

det

dets as volumes

$$= \sum_{\substack{S \subseteq |E| \\ |S|=n-1 \\ e \in S}} \|r_e\|^2 \cdot \sigma_{n-2}(\underbrace{(\Gamma \Pi_e \Gamma)}_{\text{rank } n-2}(S,S))$$

for $r_e = \Gamma(e;\cdot)$

Π_e = projection orthogonal to r_e

① symmetric matrix and $\Gamma^2 = \Gamma$

② symmetric matrix and all e-values are 0 or 1

$$= \|r_e\|^2 \cdot \sigma_{n-2}(\Gamma \Pi_e \Gamma)$$

$$= l_e \cdot 1$$

$B^T B = L$

Γ a symmetric projection: $\Gamma = B L^+ B^T$ has same e-values as $L^+ B^T B = L^+ L$ = projection onto $\mathbb{1}^\perp$

$e = (a,b)$

$$\Gamma(e,e) = \underbrace{\sqrt{w_{ab}}\,(\delta_a - \delta_b)\, L^+ (\delta_a - \delta_b)^T \sqrt{w_{ab}}}_{R_{eff}(a,b)} = \ell_e$$

$$\|\gamma_e\|^2 = \|\Gamma(e;\cdot)\|^2$$

$e = e_1$

$$\sigma_{n-1}\left(\Gamma(S;\cdot)\Gamma(\cdot;S)\right) = \det \left(S \begin{pmatrix} - & \gamma_{e_1} & - \\ - & \gamma_{e_2} & - \\ & \vdots & \\ - & \gamma_{e_{n-1}} & - \end{pmatrix} \begin{pmatrix} | & | & & | \\ \gamma_{e_1} & \gamma_{e_2} & \cdots & \gamma_{e_n} \\ | & | & & | \end{pmatrix} \right)$$

$\Gamma(S,\cdot)$

$$= \mathrm{Vol}_{n-1}\left(\gamma_{e_1}, \gamma_{e_2}, \ldots, \gamma_{e_{n-1}}\right)^2$$

Π_{e_1} = proj orthogonal to γ_{e_1}

$\Pi_{e_1} = \Pi_e$

$$= \left(\|\gamma_{e_1}\| \cdot \mathrm{vol}_{n-2}\left(\Pi_{e_1}\gamma_{e_2}, \ldots, \Pi_{e_1}\gamma_{e_{n-1}}\right)\right)^2$$

$$= \left(\|\gamma_{e_1}\| \cdot \sigma_{n-2}\left(\Gamma(S,\cdot)\Pi_e \Pi_e \Gamma(\cdot;S)\right)\right)^2$$

$$= \left(\|\gamma_{e_1}\| \cdot \sigma_{n-2}\left(\Gamma(S;\cdot)\Pi_e \Gamma(\cdot;S)\right)\right)^2$$

projection orthogonal to a vector v $= I - \dfrac{vv^T}{\|v\|^2}$