Announcements

- Start recording
- Scribe: work together, scribe to produce one set of notes
- My OH: Mon 12:30-1:30, Thu 9-10
- TC section/OH: Mon 2-3, Tue 8-9, Wed 5-6, Thu 5:30-6:30
- Fixing Problem 3.3 now extra credit
- If Zoom goes down, check Piazza
- Sync whiteboard
- Let us know if PS3 submission is still looking for partners
- Jamboard link in chat

Agenda

- Explicit Construction of Expanders
- Undirected S-T Connectivity in Logspace
- Resistor Networks
Recap: Operations on Graphs

(n, d, r) graph: n vertices, degree d, spectral expansion ≥ δ

- Squaring: (n, d, r) → (n, d² - r²)
- Tensoring: (n, d, r) → (n², d² - r²)
- Zig-zag: (n, d, r) ⊗ (d₁, d₂, r₂) → (n, d, d₁ + d₂ - r₁r₂)

(n, d, 1-δ) → (n, d², 1-δ²)
(n, d, 1-δ) → (n³, d², 1-δ²)
(n, d, 1-δ) ⊗ (d₁, d₂, 1-δ₂) → (n, d, d₁ + d₂ - r₁r₂)

Constructing Expanders

Let H = (d^4, d, 1/δ) graph (PS 3)

\[
G_0 = H^2 \quad d^2\text{-resilin}
\]

\[
G_{bn} = G_b \otimes H \quad d^2\text{-resilin} \quad n_{bn} = d^4, n_t = d^{4/6+1}
\]

\[
ω_{bn} ≤ ω_b + 2 \cdot \left(\frac{1}{δ}\right) ≤ \frac{1}{2}
\]

\[
\text{Time}(G_{bn}) = 2 \cdot \text{Time}(G_b) + O(\log n_b)
\]

= Θ(t \cdot 2^t)

not good enough

so: use tensoring too
S-T Connectivity

Given \( G = (V,E) \), \( s,t \in V \), is there a path from \( s \) to \( t \)?

- **Directed** \( G \):
  - time and space \( O(n) \) (DFS)
  - or space \( O((\log^2 n) \) (\( M^n \) via repeated squaring)
  
  \[
  M^n(s,t) = \bigoplus_{a \in V} M^{n/2}(s,a) M^{n/2}(a,t) \quad \text{depth} = O(\log n)
  \]

  **Recursive alg:** for each \( a \in V \)
  - for \( M^n(s,t) \neq 0 \), recursively check if \( M^{n/2}(s,a) \neq 0 \)
  - and if \( M^{n/2}(a,t) \neq 0 \)

  - time \( 2^{O((\log^2 n))} = n^{1+o(1)} \)

- **Undirected** \( G \):
  - in randomized space \( O(\log n) \) (PSZ 1979)
  
  Here: deterministic space \( O(\log n) \) [Reingold 2004]

So, to \( Go = G_{t-1} \)-regular,aperiodic modification of \( G \)

\( S_{t+1}, E_{t+1}, G_{t+1} = G_{t} \odot H \)

\( H \) a \((d_{1}, d_{2}, 3/4)\)-graph

\( \gamma_{t} = \frac{1}{\text{poly}(n)} \alpha \) can be cut in \( s \)

\( \gamma_{t} \alpha_{t} = (2\gamma_{t} \gamma_{t}^2) (3/4)^2 \approx \frac{18}{16} \gamma_{t} \) fit all \( \gamma_{t} \)

\( \gamma_{0}(\log n) = \Omega(1) \Rightarrow G_{0}(\log n) \alpha \text{ a constant-degree expander.} \)

Space \((G_{t+1}) = \text{Space}(G_{t}) + O(1) \Rightarrow \text{by all points of \( l_{1} \), } O(\log n) \)
describe a path of length $l$ in a degree $d$ graph = $l \cdot \log d$ bits

Digraphs $\text{w/poly}(n)$ mixing time: $\mathcal{RL}$-complete

i.e. deterministic logspace alg. for $S$-$T$ connectivity on digraphs $\text{w/poly}(n)$ mixing time $\Rightarrow$ $\mathcal{RL} = \mathcal{L}$.

open!
Resistor Networks

G = undirected, unweighted, connected graph.
View each edge as a resistor w/resistance \( r_e = \frac{1}{w_e} \)

Given voltages \( V \mapsto IR \)

Ohm's Law \((V=IR)\) says:
\[
\text{Current on edge } (a,b) = \frac{V(a)-V(b)}{r_{a,b}} = \frac{w_{a,b}}{r_{a,b}} (V(a)-V(b))
\]

\[
i = \text{Net current from } a = \sum_{b} w_{a,b} (V(a)-V(b)) = (LV)(a)
\]

Example:

\[
G
\]

\[
L = \begin{pmatrix}
5 & -1 & -1 & -3 \\
-1 & 2 & 0 & -1 \\
-1 & 0 & 2 & -1 \\
-3 & -1 & -1 & 5
\end{pmatrix}
\]

\[
\delta_A - \delta_D = \begin{pmatrix}
1 \\
0 \\
-1
\end{pmatrix}
= \begin{pmatrix}
5 & -1 & -1 & -3 \\
-1 & 2 & 0 & -1 \\
-1 & 0 & 2 & -1 \\
-3 & -1 & -1 & 5
\end{pmatrix}
\begin{pmatrix}
1 \\
7/8 \\
7/8 \\
3/4
\end{pmatrix}
\]
Flow conservation: net current at each a must equal zero.

\[ i_{\text{ext}} = L \Rightarrow L \neq 0 \text{ requires external currents} \]

\[ i_{\text{ext}} = L \text{ to realize voltages } v \]

\[ i_{\text{ext}}(a) = \text{external current entering a} \]

Note: \( i_{\text{ext}} \perp \vec{I} \), so external current conserved.

Can\(\vec{v}\) any \(i_{\text{ext}} \in \text{Im}(L)\), \(v = L^{*}i_{\text{ext}}\) are induced voltages

\[ \vec{v} \perp \vec{I} \quad \text{PS3} \]

up to adding mult. at \( \vec{I} \)
Effective Resistance between \( a \) and \( b \):

\[
R^{eff}(a, b) = \begin{cases} v(a) - v(b) & \text{when } i_{ext} = e_a - e_b \\ \left( L^+ i_{ext} \right)(a) - \left( L^+ i_{ext} \right)(b) \end{cases}
\]

\[
= e_a - e_b^T L^+ (e_a - e_b)
\]

\[
= \| L^{1/2} e_a - L^{1/2} e_b \|^2
\]

\( M \) is PSD with eigenvectors \( \lambda_1, \ldots, \lambda_n \Rightarrow M^{1/2} \) has eigenvalues \( \lambda_1^{1/2}, \ldots, \lambda_n^{1/2} \)

**Thm:**

\[
\frac{1}{R^{eff}(a, b)} = \min_{x \in \mathbb{R}^n \text{ s.t.} x(a) = 1, x(b) = 0} \left( x^T L x \right)
\]

**Proof sketch:** Let \( i_{ext} = e_a - e_b \)

Claim: \( x \) is unique

\[
V = L^+ i_{ext}
\]

\[
x = \frac{V - v(b)1}{v(a) - v(b)}
\]

\[
x(a) = \frac{v(a) - v(b)}{R^{eff}(a, b)} > 1
\]

\[ f(x) = x^T L x \]
Cor: If we decrease resistances (= increase edge weights), effective resistances can't increase.

(Rayleigh's Monotonicity Principle)

Exercise for Breakouts

Derive effective resistances between a and b in the following networks by finding voltages \( \mathbf{V} \) inducing current \( \mathbf{i}_{\text{ext}} = \delta_a - \delta_b \) (i.e. \( \mathbf{L}\mathbf{V} = \delta_a - \delta_b \)).