Announcements

- start recording
- scribe: work w/other scribe to produce one set of notes
- My OH: Mon 12:30–1:30, Thu 9–10
- TF hybrid section/OH: Mon 2–3, Tue 8–9, Wed 5–6, Thu 5:30–6:30
- PS 3 posted
- If Zoom goes down, check Piazza
- Sync whiteboard
- Post on Piazza by tomorrow to find project partners
- No jamboard or gather today

Agenda

- Expansion of random graphs
- Random walks on expanders
- Operations on expanders

Measures of Expansion

Thm: Let $\mathcal{G}$ be an infinite family of regular, undirected, lazy, constant-degree graphs

The following are equivalent

1) $\exists\ \gamma > 0$ s.t. every $G \in \mathcal{G}$ has spectral expansion at least γ. (i.e. $\omega(G) = \min\{\omega_2, -\omega_n\} \leq 1 - \gamma$)

$d = O(1)$ lazy

2) $\exists\ \varepsilon > 0$ s.t. every $G \in \mathcal{G}$ is an $(n/2, \varepsilon)$ edge expander. (i.e. $\forall S\ |S| \leq n/2,\ |e(S, S^c)| \geq \varepsilon \cdot d \cdot |S|$)

$d = O(1)$ lazy

3) $\exists\ \delta > 0$ s.t. every $G \in \mathcal{G}$ is an $(n/2, 1+\delta)$ vertex expander. (i.e. $\forall S\ |S| \leq n/2\ \ |N(S)| \geq (1+\delta) \cdot |S|$)

Expander Mixing Lemma + Converse

G regular digraph w/ spectral expansion $\gamma = 1 - \omega$

$$\theta = \omega \ \downarrow \quad \uparrow\ \omega = O(\theta \log(\tfrac{1}{\theta}))$$

$\forall\ S, T \subseteq V$ w/ $|S| = \alpha n$, $|T| = \beta n$

$$\left| \frac{e(S,T)}{|E|} - \alpha\beta \right| \leq \theta \sqrt{\alpha \cdot (1-\alpha) \cdot \beta \cdot (1-\beta)}$$

Existence of Expanders

"Probabilistic Method"

A uniformly random d-regular graph is a very good expander whp.

e.g.

<u>Vertex Expansion</u>: $(\alpha_{d,\varepsilon} n, \underset{\sim}{d-1-\varepsilon})$ - vertex expander

vertex expansion of ∞ d-regular tree

$\alpha_{d,\varepsilon} > 0$

$(n/2, 1+\delta_d)$ - vertex expander

$\delta_d > 0$

$\delta_d \to 1$ as $d \to \infty$

<u>Pf idea</u>:
- For each set S of size k argue $\Pr_G[S \text{ does not expand enough}] \ll \frac{1}{\binom{n}{k}}$
- Union bound over sets S

<u>Spectral Expansion</u>: $\omega = \frac{2\sqrt{d-1}}{d} + \varepsilon$

[Conj. by Alon '86, Proof by Friedman '04]

largest e-value of ∞ d-regular tree

<u>Pf idea</u>:
- $Tr(\underline{W}^{2t}) = \sum_{j=1}^{n} \omega_j^{2t} \geq 1 + \omega(G)^{2t}$
- $\Pr_G[\omega(G) \geq \omega] \leq \frac{\mathbb{E}_G[Tr(W^{2t})] - 1}{\omega^{2t}}$
- $Tr(W^{2t}) = \sum_{a=1}^{n} W^{2t}_{a,a} = \Pr_{\substack{\text{r.w.} \\ \text{of length} \\ 2t}}\left[\begin{matrix}\text{walk starts} \\ \text{and ends at } a\end{matrix}\right]$

Goal: show $\Pr_{G, \text{r.w.}}\left[\begin{array}{l}\text{r.w. of length } 2t \\ \text{starts + ends at } a\end{array}\right] \leq \frac{1}{n} + \frac{1}{n^{1+c}}$

for $t = O(\log n)$

See Spielman Ch. 8 for random dense graphs. $G(n,p)$

Ramanujan Graphs: $\omega \leq \frac{2\sqrt{d-1}}{d}$ [no ε!]

- Not known that random graphs have this whp.
- Explicit constructions from deep number theory (relying on proven "Ramanujan Conjectures")
- Bipartite Ramanujan graphs recently proved (2015+) to exist using probabilistic argument that only establishes $\Pr > 0$

[see Spielman Part VII]

1

Random Walks on Expanders

Motivating example: Power Method

M psd w/ largest e-value μ_1

① choose $x \xleftarrow{R} \{\pm 1\}^n$

② output $y = M^k x$ for $k = O\left(\frac{\log(n/\varepsilon)}{\varepsilon}\right)$

w.p. $\geq 3/16$ over x,

$$\frac{y^T M y}{y^T y} \geq (1-\varepsilon) \cdot \mu_1$$

Reducing failure probability

- Repeat t times w/ $x^{(1)}, \ldots, x^{(t)} \xleftarrow{R} \{\pm 1\}^n$
- Compute $y^{(1)} = M^k x^{(1)}, \ldots, y^{(t)} = M^k x^{(t)}$
- Output $y = y^{(i)}$ maximizing $\frac{y^{(i)} M y^{(i)}}{(y^{(i)})^T (y^{(i)})}$

$$\Pr\left[\frac{y^T M y}{y^T y} < (1-\varepsilon)\mu_1\right] \leq \left(\frac{13}{16}\right)^t$$

$$= 2^{-\Omega(n)}$$

for $t = O(n)$

$\#$ random bits used $= t \cdot n = O(n^2)$

Can we do better?

- choose $x^{(1)}, \ldots, x^{(t)}$ using a random walk on an expander $G=(V,E)$

$N = |V| = 2^n \qquad V \Leftrightarrow \{\pm 1\}^n$

- choose $x^{(1)} \xleftarrow{R} V$

$x^{(2)} \xleftarrow{R} \{x^{(1)}\text{'s } d \text{ neighbors}\}$

$x^{(3)} \xleftarrow{R} \{x^{(2)}\text{'s } d \text{ neighbors}\}$

$\vdots$

$x^{(t)} \xleftarrow{R} \{x^{(t-1)}\text{'s } d \text{ neighbors}\}$

$\#$ random bits $= n + O(t \log d)$

$= O(n)$

$\Uparrow$

$t = O(n)$, $d = O(1)$

$= O(\log N)$

Does error still reduce?

$$B = \left\{ x \in \{\pm 1\}^n : \text{for } y = M^k x \quad \frac{y^T M y}{y^T y} < (1-\varepsilon)\cdot \mu_1 \right\}$$

$$\mu = \mu(B) = \frac{|B|}{|V|} \leq \frac{13}{16}$$

<u>Thm</u>: If G has spectral expansion $\gamma = 1 - \omega$ and $V_1, \ldots, V_t$ are a random walk on G w/ uniform start vertex V_1, then

$\forall B$

$$\Pr\left[\underbrace{\bigwedge_{i=1}^{t} (V_i \in B)}_{V_1 \in B \,\wedge\, V_2 \in B \,\wedge \cdots \wedge\, V_t \in B}\right] \leq \left(\overline{\mu + \omega \cdot (1-\mu)}\right)^t$$

$$\leq 2^{-\Omega(t)} \text{ for constants } \mu, \omega < 1$$

Proof:

W = random walk matrix

$$P = \operatorname{diag}(\vec{1}_B) = \begin{array}{c} \\ B\{ \end{array} \begin{pmatrix} \overbrace{1 \; 1}^{B} & & 0 \\ 0 & 0 & \ddots \end{pmatrix}_{N}$$

$u = \vec{1}/N$

$|v|_1 = \sum_{j=1} |v_j|$

$\|v\|_2 = \|v\| = \sqrt{\sum_{j=1}^{n} v_j^2}$

$$\Pr\left[\bigwedge_{j=1}^{t} V_j \in B\right] = \left| (PW)^{t-1} P u \right|_1$$

$$= \left| (PWP)^{t-1} P u \right|_1 \quad (P^2 = P)$$

$$\leq \sqrt{N} \cdot \|(PWP)^{t-1} P u\|$$

$$\underset{C\text{-}S}{\leq} \sqrt{N} \cdot \underbrace{\|PWP\|}_{\downarrow}^{t-1} \|Pu\|$$

$$\leq \sqrt{N} \cdot \left(\mu + \omega \cdot (1-\mu)\right)^{t} \cdot \sqrt{\frac{\mu}{N}}$$

Def (spectral norm): $\|M\| = \max_{x \neq 0} \frac{\|Mx\|}{\|x\|}$ = largest singular value of M

Matrix Decomposition

Lemma: G has spectral expansion γ iff

$$W = \gamma J + (1-\gamma) E$$

where J = all $1/n$ matrix

and $\|E\| \leq 1$

"good expanders (when $\gamma \approx 1$) are good approximations of the complete graph" (despite being sparse)

$$P = \mathrm{diag}(\vec{1}_B)$$

Thus:

$$\|PWP\| = \|\gamma PJP + \underline{(1-\gamma)PEP}\|$$

$$\leq \gamma \cdot \|PJP\| + (1-\gamma) \cdot \underline{\|PEP\|}$$

$$\leq \gamma \cdot \mu + (1-\gamma) \cdot 1$$

$$= \mu + \omega \cdot (1-\mu)$$

$$\underline{PJPx = \left(\sum_{j \in B} x_j\right) \cdot \frac{\vec{1}_B}{N}}$$

There is also a Chernoff bound for expander walks $\rightarrow$ randomness-efficient error reduction for randomized algs w/ 2-sided error.

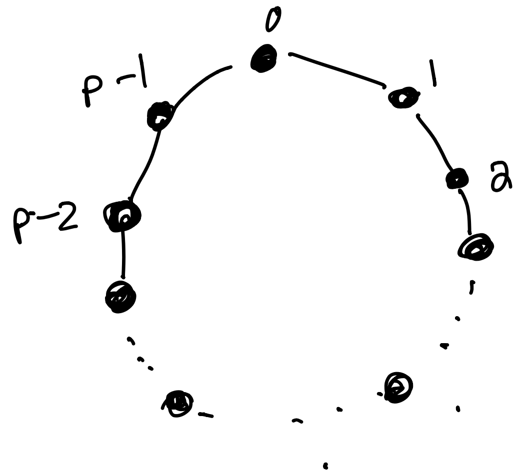
Explicit Constructions of Expanders

Goal: infinite family of graphs $\mathcal{G} = \{G_i\}$ s.t.

- $\exists$ constant d s.t. every G_i is d-regular
- $\exists$ constant $\gamma > 0$ s.t. every G_i has spectral expansion at least γ
- given $a \in \{1, \dots, n_i\}$ and $j \in \{1, \dots, d\}$ can compute j^{th} neighbor of vertex a in G_i in time poly($\log n_i$)
- the family $\{n_i\}$ of sizes is not too sparse ($\rightarrow$ can convert into a family of expanders of all sizes)

$n_i = \#$ vertices in G_i

Example:



p vertices
for a prime p

- Proof of expansion: deep number theory

Our Approach:

- start w/a "constant-sized expander"
  eg from ps3 problem 4
- repeatedly apply graph ops to
  get larger expanders

(n, d, γ)-graph: n vertices, degree d, spectral expansion $\geq \gamma$

<u>Squaring</u>: $(n, d, \gamma) \longmapsto (n, d^2, \gamma^2)$

<u>Tensoring</u>: $(n, d, \gamma) \longmapsto (n^2, d^2, \gamma)$

$G_1 \otimes G_2$: • vertex set $V = V_1 \times V_2$

• edge weights

$$w_{(a_1,a_2),(b_1,b_2)} = w_{a_1 b_1} w_{a_2 b_2}$$

• adjacency matrix $M_1 \otimes M_2$

• random-walk matrix $W_1 \otimes W_2$

evalues/ singular values $\alpha_1 \alpha_2$ s.t.

α_1 evalue of W_1, α_2 e-value of W_2