Announcements

- Start recording
- Scribe: write 4-6 scribes to produce one set of notes
- My OH: Mon 12:30-1:30, Thu 9-10
- TC Hybrid section: Mon 2-3, Tue 8-9, Wed 5-6, Thu 5:30-6:30
- PS 3 posted
- If Zoom goes down, check Piazza
- Sync whiteboard
- Post on Piazza by tomorrow to find project partners
- No Jamboard or emails today

Agenda

- Expansion of random graphs
- Random walks on expanders
- Operations on expanders
Measures of Expansion

Claim: Let $N$ be an infinite family of regular, undirected, binary, constant-degree graphs.

The following are equivalent:
1) $\exists \epsilon > 0$ s.t. every $G \in N$ has spectral expansion at least $\epsilon$, i.e., $\varnothing(G) = \min \{\lambda_2(G), -\lambda_3(G)\} \leq 1-\epsilon$

2) $\exists \epsilon > 0$ s.t. every $G \in N$ is an $(n/2, \epsilon)$-expander, i.e., $\forall S \subseteq V$, $|S| \leq n/2$, $|\partial(S)| \geq \epsilon \cdot |S|$

3) $\exists \epsilon > 0$ s.t. every $G \in N$ is an $(n/2, \epsilon)$-regular expander, i.e., $\forall S \subseteq V$, $|S| \leq n/2$, $|\partial(S)| \geq (1+\epsilon) \cdot |S|$

Exander Mixing Lemma + Converse

If a regular graph $G$ has spectral expansion $\mu = 1-\omega$

Then $\omega = O(\sqrt{\mu} \log(1/\delta))$

$\forall S \subseteq V$, $|S| \leq n$, $\Pi = S \cup \partial(S)$

$|e(S,T)| / |S| \leq \omega \leq \Theta \sqrt{n \cdot (1-\omega) \cdot \beta \cdot (\gamma - \beta)}$
Existence of Expanders

A uniformly random \( d \)-regular graph is a very good expander w.h.p.

- Vertex expansion of \( d \)-regular tree
  \( \lambda_d = \frac{d}{d-1} \)
  - vertex expansion
  \( \lambda_d > 0 \)
  \( \left( \frac{n}{2}, 1 + \delta_d \right) \)
  - vertex expansion
  \( \delta_d > 0 \)
  \( \delta_d \to 1 \) as \( d \to \infty \)

Proof idea:
- For each set \( S \) of size \( k \)
  - \( \Pr \left[ S \text{ does not expand much} \right] \leq \frac{1}{(n/k)} \)
- Union bound over sets \( S \)

Spectral Expansion:
\[
\omega = \frac{2\sqrt{d-1}}{d} + \epsilon
\]

- Largest eigenvalue of \( d \)-regular tree

Proof idea:
- \( \text{Tr} \left( W^2b \right) = \sum_{j=1}^{\infty} w_{j}^{2b} \geq 1 + \omega(6)^{2b} \)

- \( \Pr \left[ \omega(6) \geq \omega \right] \leq \frac{6}{\epsilon} \left[ \text{Tr} \left( W^{2b} \right) - 1 \right] \)

- \( \text{Tr} \left( W^{2b} \right) = \sum_{a=1}^{c} W_{a,a}^{2b} = \Pr \left[ \text{walk starts at } u, \text{ ends at } v \right] \quad \text{c.u. at } v \geq 2\delta \)
Goal: show \( P[ \text{rw. at least } 2t \text{ starts from } a] \leq \frac{1}{n} + \frac{1}{n^{0.5}} \)
for \( t = O(\log n) \)

See Spielman Ch. 8 for random dense graphs. \( G(n, p) \)

Ramanujan Graphs: \( \omega \leq \frac{2 \sqrt{d-1}}{d} \) \[ \text{[no exp!] \]}

- Not known that random graphs have this whp.
- Explicit constructions from deep number theory (relying on proven "Ramanujan Conjectures")
- Bipartite Ramanujan graphs recently proved (2015+) to exist using probabilistic argument that only establishes \( R > 0 \)

[see Spielman Part VII]
Random Walks on Expanders

Motivating example: Power Method

\[ M \text{ psd } \implies \text{largest eigenvalue } \lambda. \]

1. choose \( x \triangleq \mathcal{E}(n) \)
2. output \( y = M^k x \quad \text{for } k = O\left( \frac{\log(n/\varepsilon)}{\varepsilon^2} \right) \)

w.p. \( \geq \frac{3}{16} \) on \( x \),

\[ \frac{y^T M y}{y^T y} \geq (1-\varepsilon) \cdot \lambda. \]

Reducing failure probability

- Repeat \( t \) times w/ \( x^{(1)}, \ldots, x^{(t)} \)
- Compute \( y^{(1)} = M^k x^{(1)}, \ldots, y^{(t)} = M^k x^{(t)} \)
- Output \( y = y^{(t)} \) maximizing \( \frac{y^{(t)^T M y^{(t)}}}{(y^{(t)})^T y^{(t)}} \)
\[ \Pr \left[ \frac{y^T M y}{y^T y} < (1-\varepsilon) \mu_1 \right] \leq \left( \frac{13}{16} \right)^t \]

\[ = 2^{-\Omega(n)} \]

\[ \text{for } t = O(n) \]

# random bits used = \( t \cdot n = O(n^2) \)

Can we do better?

- choose \( x^{(1)} \), ..., \( x^{(t)} \) via a random walk on an expander \( G = (V, E) \)

\[ N = |V| = 2^n \]

\[ V \rightarrow 3 \pm 13^n \]

- choose \( x^{(1)} \) \( \in \) \( V \)

\[ x^{(2)} \leftarrow \{ x^{(1)} \text{'s d neighbours} \} \]

\[ x^{(13)} \leftarrow \frac{3}{4} x^{(2)} \text{'s d neighbours} \]

\[ : \]

\[ x^{(t)} \leftarrow \frac{3}{4} x^{(t-1)} \text{'s d neighbours} \]
# Random bits = \( n + O(t \log d) \)

\[ \approx O(n) \]

\[ t = O(d) \quad d = O(1) \]

Does error still reduce?

\[ B = \sum_{i=1}^{X} \mathbb{1}_{x \in B^n} : \text{for } y = M^k \quad \frac{y^T M y}{y^T y} < (1-\epsilon) x_i \]

\[ u = u(B) = \frac{|B|}{|V|} \leq \frac{13}{16} \]

Thm: If \( G \) has spectral expansion \( \kappa = 1 - \omega \) and \( V_1, \ldots, V_b \) are a random walk on \( B \) on \( G \) with uniform start vertex \( V \), then

\[ \Pr \left[ \bigwedge_{i=1}^{b} \left( N_i \in B \right) \right] \leq \left( u + \omega \cdot (1-u) \right)^b \]

\[ V_1 \epsilon B \land V_2 \epsilon B \land \ldots \land V_b \epsilon B \leq 2^{-\Omega(b)} \text{ for constants } u, \omega < 1 \]
Proof:

\[ W = \text{random walk matrix} \]
\[ P = \text{diag } (I_B) = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \]
\[ u = \frac{1}{\sqrt{N}} \]

\[ P^t \left[ \sum_{i=1}^t (N \cdot \sigma B) \right] = \left\| (PW)^t Pu \right\|_1 \leq \frac{1}{\sqrt{N}} \left\| (PW)^t Pu \right\|_2 \leq \frac{1}{\sqrt{N}} \|Pu\| \leq \sqrt{N} \cdot \|PW\|^{-b} \|Pu\| \leq \sqrt{N} \cdot (n + \omega \cdot (1 - n))^b \cdot \sqrt{N} \]

Def (spectral norm):

\[ \|M\| = \max_{x \neq 0} \frac{\|Mx\|}{\|x\|} = \text{largest singular value of } M \]

Matrix Decomposition

Lemma: \( G \) has spectral expansion \( \lambda, \mu \)

\[ W = \lambda J + (1 - \lambda) E \]

where \( J = \text{all } 1 \text{ matrix} \)

and \( \|E\| \leq 1 \)
Thus: \[ \| PWP \| = \| \frac{\gamma PJP + (1-\gamma)PEP}{\gamma (PJP) + (1-\gamma) EPE} \| \leq \gamma \cdot \| PJP \| + (1-\gamma) \cdot 1 \]

\[ = \mu + \omega \cdot (1-\mu) \]

\[ PJP = \left( \sum_{J \in X} x_J \right) \cdot \frac{1_B}{N} \]

There is also a Chernoff bound for
exponential walks \rightarrow randomness - efficiency
time reduction for randomized algorithms
w/2-sided errors.
Explicit Constructions of Expanders

Goal: intrinsic family of graphs $\mathcal{G} = \{G_i\}_i$ s.t.

- $d$ is a constant s.t. each $G_i$ is $d$-regular
- For all $\gamma > 0$ s.t. each $G_i$ has spectral expn $\gamma$

- Given $a \in E_1, \ldots, n_i \in \mathbb{Z}$ and $j \in E_1, \ldots, d_i$
  
  can compute $j^{th}$ neighbor of vertex $a$ in $G_i$
  in time poly($\log n_i$)

- The family $\{n_i\}$ of sizes is not too sparse

  $\implies$ can convert into a family of expanders of all sizes

$n_i = \#$ vertices in $G_i$
Example:

- Root of expasion: deep number theory

Our Approach:

- Start w/ a "constant-sized expander"  
  eg from ps3 problem 4
- Repeatedly apply graph ops to  
  get larger expanders
\((n, d, \lambda)\) - graph: \(n\) vertices, degree \(d\), spectral expansion \(\geq \lambda\)

Squaring: \((n, d, \lambda) \rightarrow (n, d^2, \lambda^2)\)

Tensoring: \((n, d, \lambda) \rightarrow (n^2, d^2, \lambda^2)\)

\(G_1 \otimes G_2\) : vertex set \(V = V_1 \times V_2\)

edge weights

\[ W \quad = \quad W_{a_1 b_1} W_{a_2 b_2} \]

\((a_1, a_2), (b_1, b_2)\)

- adjacency matrix \(M_1 \otimes M_2\)
- random-walk matrix \(W_1 \otimes W_2\)

eigenvalues \(\alpha, \alpha_2\) s.t.

\(\alpha_1\) eigen of \(W_1\), \(\alpha_2\) e-val of \(W_2\)