Announcements

- start recording
- scribe: work w/other scribe to produce one set of notes
- My OH: ~~Mon 12:30-1:30~~ TODAY 4:30-5:30, Thu 9-10
- TF hybrid section/OH: Mon 2-3, Tue 8-9, Wed 5-6, Thu 5:30-6:30
- PS 3 posted
- If Zoom goes down, check Piazza
- Sync whiteboard
- Final project ideas feedback by tomorrow, use Piazza to find partners
- Jamboard link in chat
- Discuss feedback
- No gather today

Agenda

- Vertex expansion
- Expander mixing lemma
- Expansion of random graphs
- Random walks on expanders

Expander Graphs

Many applications in TCS!

d-regular n-vertex, unweighted digraphs, $n \to \infty$

SPARSE: typically $d = O(1)$

(sometimes $d = \text{polylog}(n)$)

more standard notation: $\omega_\neq$

WELL-CONNECTED: several defs

1) spectral expansion γ : $\gamma(G) = 1 - \omega(G) \geq \gamma$ $\in [0,1]$

$$\omega(G) \overset{\text{def}}{=} \omega_u(G) = \max_p \frac{\|W_p - u\|}{\|p - u\|} = \max_{x \perp u} \frac{\|Wx\|}{\|x\|} = \max\{\omega_2, -\omega_n\}$$

G undirected

u = uniform distribution $= \vec{1}/n$

Typically want $\gamma > 0$ independent of n.

Ideally maximize $\gamma = 1 - \omega$ as a function of d

2) (k, a) vertex expansion

$\forall\, S$ $\quad |S| \le k, \quad |N(S)| \ge a \cdot |S|$

$N(S) = \{b : \exists a \in S \ (a,b) \in E\}$

Typically $k = \Omega(n)$, $a = 1 + \Omega(1)$

Ideally maximize a as fnctn of n, k, d.

$a \ge 1$ by regularity

e.g. $a = d - 1.01$

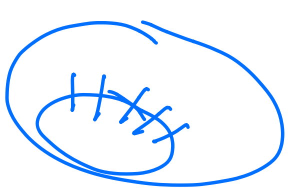
3) (k, ε) edge expansion

$\forall\, S$ $\quad |S| \le k, \quad |e(S, S^c)| \ge \varepsilon \cdot d \cdot |S|$

Typically $k = \Omega(n)$, $\varepsilon = \Omega(1)$

Ideally maximize ε as func. of n, k, d

$$\phi(S) = \frac{|e(S, S^c)|}{d \cdot |S|}$$



Edge Expansion vs. Spectral Expansion

Thm: Let $\mathcal{G}$ be an infinite family of regular undirected, lazy graphs,

The following are equivalent

1) $\exists\ \gamma > 0$ s.t. every $G \in \mathcal{G}$ has spectral expansion at least γ.

$\Downarrow$ w/o laziness

$\Uparrow$ laziness necessary

2) $\exists\ \varepsilon > 0$ s.t. every $G \in \mathcal{G}$ is an $(N/2, \varepsilon)$ edge expander.

Proof: By Cheeger's Inequalities

Vertex Expansion vs. Spectral Expansion

Thm: Let $\mathcal{G}$ be a family of constant-degree, regular digraphs. TFAE:

1) $\exists\ \gamma > 0$ s.t. every $G \in \mathcal{G}$ has spectral expansion at least γ.

$\Downarrow$ even for unbounded degree $\qquad \Uparrow$ $d = O(1)$

2) $\exists\ \varepsilon > 0$ s.t. every $G \in \mathcal{G}$ is an $(N/2, 1+\varepsilon)$ vertex expander.

"family of constant-degree expanders"

Proof: $(2) \Rightarrow (1)$ in section

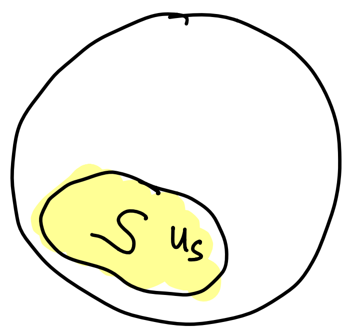
$(1) \Rightarrow (2)$: Let $S \subseteq V$ $\quad |S| = \alpha n \quad \alpha \le 1/2$

$$|N(S)| = |\mathrm{Supp}(W u_S)| \ge \frac{1}{CP(W u_S)}$$

$$CP(p) = \Pr_{\substack{X, X' \sim p \\ \text{indep}}}[X = X'] = \sum_a p_a^2 = \|p\|^2$$

$$= \|u\|^2 + \|p - u\|^2$$

$$= \frac{1}{n} + \|p - u\|^2$$



(*) $$\left(CP(W u_S) - \frac{1}{n}\right) \le \omega(G)^2 \cdot \left(CP(u_S) - \frac{1}{n}\right)$$

$$\frac{1}{|N(S)|} - \frac{1}{n} \le (1-\gamma)^2 \cdot \left(\frac{1}{|S|} - \frac{1}{n}\right)$$

Exercise for Breakouts

M = adjacency matrix of a constant-degree expander on n vertices

Q: which expansion properties are <u>retained</u> by following graphs?

Adjacency Matrix	Spectral Expansion $\Omega(1)$?	$(n^{1/2}, \Omega(1))$ edge expansion?	$(n^{1/2}, 1+\Omega(1))$ vertex expansion?	$(n^{1/3}, \Omega(1))$ edge expansion?	$(n^{1/3}, 1+\Omega(1))$ vertex expansion?
M^2	✓				
$M + (\log n) I$	$\omega \geq 1 - d/\log n$ ✗	✗	✓	✗	✓
$\begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}$	✗	✗	✗	✓	✓
$\begin{pmatrix} 0 & M \\ M & 0 \end{pmatrix}$	✗	✓	✗	✓	✓
$\begin{pmatrix} M & M \\ M & M \end{pmatrix}$	✓	✓	✓	✓	✓

Expander Mixing Lemma

Let G be a regular digraph w/ spectral expansion $\gamma = 1-\omega$

Then for every $S, T \subseteq V$ with $|S| = \alpha n$, $|T| = \beta n$

$$\left| \frac{|e(S,T)|}{|E|} - \alpha\beta \right| \le \omega \cdot \sqrt{\alpha \cdot (1-\alpha) \cdot \beta \cdot (1-\beta)} \qquad (*)$$

edge expansion corresponds to $T = S^c$ and one direction of inequality

Proof: $|e(S,T)| = \chi_T^T M \chi_S$ $\qquad M = $ adj matrix

$$\frac{|e(S,T)|}{|E|} = \frac{\chi_T^T W \chi_S}{n} \qquad W = \text{r.w. matrix}$$

expand into 4 terms

$\chi_T = \beta n \vec{u} + \chi_T^\perp$

$\chi_S = \alpha n \vec{u} + \chi_S^\perp$

cross-terms zero

gives $\alpha\beta$ $\qquad W\vec{u} = \vec{u}$

$$\left| \chi_T^\perp W \chi_S^\perp \right| \le \|\chi_T^\perp\| \cdot \|W \chi_S^\perp\|$$

$$\le \omega \cdot \|\chi_T^\perp\| \cdot \|\chi_S^\perp\|$$

Converse: satisfying $(*)$ for all S, T implies

$$\omega(G) = O(\omega \log(1/\omega)) = \tilde{O}(\omega)$$

Existence of Expanders

A uniformly random d-regular graph is a very good expander whp.

e.g.

<u>Vertex Expansion</u>: $(\alpha_{d,\varepsilon} \cdot n, \overset{\sim}{d-1-\varepsilon})$ - vertex expander

vertex expansion of ∞ d-regular tree

$\alpha_{d,\varepsilon} > 0$

$(n/2, 1+\delta_d)$ - vertex expander

$\delta_d > 0$

$\delta_d \to 1$ as $d \to \infty$

<u>Pf idea:</u>

<u>Spectral Expansion</u>: $\omega = \frac{2\sqrt{d-1}}{d} + \epsilon$

[Conj. by Alon '86, Proof by Friedman '04]

largest e-value of ∞ d-regular tree

<u>Pf idea:</u> · $Tr(W^{2t}) \geq$

· $\Pr_G[\omega(G) \geq \omega] \leq$

· $Tr(W^{2t}) =$

Goal: show $\Pr_{G,\text{r.w.}}\left[\begin{array}{l}\text{r.w. of length } 2t \\ \text{starts + ends at } a\end{array}\right] \le \frac{1}{n} + \frac{1}{n^{1+c}}$

for $t = O(\log n)$

See Spielman Ch. 8 for random dense graphs. $G(n,p)$

Ramanujan Graphs: $\omega \le \frac{2\sqrt{d-1}}{d}$ [no ε!]

- Not known that random graphs have this whp.
- Explicit constructions from deep number theory (relying on proven "Ramanujan Conjectures")
- Bipartite Ramanujan graphs recently proved (2015+) to exist using probabilistic argument that only establishes $\Pr > 0$

[see Spielman Part VII]

Random Walks on Expanders

Motivating example: Power Method