Announcements

- start recording
- scribe: work w/other scribe to produce one set of notes
- My OH: Mon 12:30-1:30, Thu 9-10
- TF section/OH: Mon 2-3, Tue 8-9, Wed 5-6, Thu 5:30-6:30
- PS 4 posted. Project proposal due Sun 11/8, problems due TUE 11/17
- If Zoom goes down, check Piazza
- Sync whiteboard
- Post your project topic on spreadsheet
- Sign up for 15min check in with me this week or next
- Jamboard link in chat

Agenda

- Loewner order
- Comparing graphs
- Expanders + spectral sparsifiers

Loewner Order: for $n \times n$ symmetric matrices A, B

if apply to A^TA & B^TB

$A \succeq B$ means $A - B$ is psd $\iff \|Av\|^2 \geq \|Bv\|^2$

i.e. $v^TAv \geq v^TBv$ for all $v \in \mathbb{R}^n$

only depends $(A+A^T)/2$

if B psd implies $\ker(A) \subseteq \ker(B)$

Example: $L_G \succeq L_H$ if H is a subgraph of G (more generally if $w_G(e) \geq w_H(e)$ for all e)

$$x^T L_G x = \sum_{(a,b) \in E} w_{(a,b)} (x(a) - x(b))^2$$

sometimes write $G \succeq H$

Prop: if $A \succeq B$ then $\lambda_k(A) \geq \lambda_k(B)$ for $k = 1, \dots, n$

Pf: Courant-Fischer

Exercise for Breakouts

True or False? For all symmetric $n \times n$ matrices A, B

1) $A \succeq B \Rightarrow CAC^T \succeq CBC^T$ for all $n \times n$ C

True $\quad v^T CAC^T v = (C^Tv)^T A (C^Tv)$

Note: this does not hold for CAC^{-1}

2) $\lambda_k(A) \geq \lambda_k(B)$ for all $k \Rightarrow A \succeq B$

False e.g. $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ (True if A & B have common eigenbasis and λ_k = corresponding eigenvalue)

3) $\|A\| \leq \varepsilon \Rightarrow A \preceq \varepsilon \cdot I$

True $\quad x^TAx \leq \|A\| \cdot \|x\|^2 \leq \varepsilon \cdot x^T I x \Rightarrow A \preceq \varepsilon \cdot I$

$\|A\| = \max_i |\lambda_i|$

4) $A \preceq \varepsilon I \Rightarrow \|A\| \leq \varepsilon$

$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \preceq \begin{pmatrix} \varepsilon & & 0 \\ & \ddots & \\ 0 & & \varepsilon \end{pmatrix}$

A after diagonalizing

False $\quad A = \begin{pmatrix} -2\varepsilon & 0 \\ 0 & -2\varepsilon \end{pmatrix}$

$$\|A\| \le \varepsilon \iff -\varepsilon I \preceq A \preceq \varepsilon I$$

Thm (ps 1): for every undirected, connected G of diameter $\le r$, $\boxed{\lambda_2(G) \ge \frac{1}{O(r\cdot n)}}$

Pf: idea: compare L_G w/ L_{K_n}

$$L_{K_n} = \sum_{\{a,b\}\subseteq V} L_{(a,b)}$$

"path inequality"

$$\preceq \sum_{\{a,b\}\subseteq V} \underbrace{|P_{(a,b)}|}_{\text{length of path}} \cdot L_{P_{(a,b)}}$$

("Path inequality" $P_{(a,b)}$ is a path in G between a and b)

$$\preceq \binom{n}{2} \cdot r \cdot L_G \quad (P_{(a,b)} \text{ a subgraph of } G)$$

$$\lambda_2(G) \ge \frac{\lambda_2(K_n)}{\binom{n}{2}\cdot r} = \frac{n}{\binom{n}{2}\cdot r} = \frac{2}{(n-1)\cdot r}$$

Path Inequality: If P is a path of length r between vertices a, b, then $L_{(a,b)} \preceq r \cdot L_P$

Proof: $a=0 \xrightarrow{w_1} 1 \xrightarrow{w_2} 2 \;—\; \cdots \;—\; \xrightarrow{w_r} b=r$

$$x^T L_P x = \sum_{i=1}^{r} (x(i)-x(i-1))^2 \cdot w_i$$

$$x^T L_{(a,b)} x = (x(r)-x(0))^2 = \left(\sum_{i=1}^{r} (x(i)-x(i-1))\right)^2$$

$$\le \left(\sum_{i=1}^{r} \left(\sqrt{w_i}(x(i)-x(i-1))\right)^2\right) \cdot \underbrace{\left(\sum_{i=1}^{r} \left(\frac{1}{\sqrt{w_i}}\right)^2\right)}_{r}$$

$$= r \cdot x^T L_P x$$

Weighted Path Inequality.

If P_w has edge weights $w_1, w_2, \ldots, w_r$

then $L_{(a,b)} \preceq \left(\sum_{i=1}^{r} \frac{1}{w_i}\right) \cdot L_{P_w}$

Thm: Let G be a regular undirected graph with r.w. matrix W

TFAE (the following are equivalent)

1) $\gamma(G) \geq 1-\omega$

2) $-\omega I \preceq W - J \preceq \omega I$ J = all $1/n$ matrix

3) $(1-\omega)(I-J) \preceq I - W \preceq (1+\omega)(I-J)$

Proof: $(1 \Leftrightarrow 2)$ $-wI \preceq W - J \preceq \omega I \iff \|W-J\| \leq \omega$

$[(W-J)x = Wx^{\perp}]$

$\iff \forall x \perp \vec{1} \quad \|Wx\| \leq \omega \cdot \|x\|$

$(1 \Leftrightarrow 3)$ diagonalize $I-W$ and $I-J$ w/eigenbasis for W

eigenvalues of $I-J$: 0 w/multiplicity 1

1 w/ multiplicity $n-1$

(3) says $(1-\omega)\cdot 1 \leq 1-\omega_i \leq (1+\omega)\cdot 1$ for $i=2,\dots,n$

i.e. $|\omega_i| \leq \omega$

Corollary: For every $\varepsilon > 0$ $\exists$ c such that $\forall n$ there is a graph H w/ $\leq cn$ edges s.t.

$$(1-\varepsilon) L_{K_n} \preceq L_H \preceq (1+\varepsilon) L_{K_n}$$

Next time: $\forall G$ $\exists H$ w/ $O\left(\frac{n \log n}{\varepsilon^2}\right)$ edges

s.t. $(1-\varepsilon) L_G \preceq L_H \preceq (1+\varepsilon) L_G$

"H is a spectral sparsifier of G"

Stronger than a cut sparsifier:

$$(1-\varepsilon)\phi_G(S) \leq \phi_H(S) \leq (1+\varepsilon)\phi_G(S)$$

Variants:
- $(1+\varepsilon)^{-1} \cdot L_G \preceq L_H \preceq (1+\varepsilon) \cdot L_G$
- $e^{-\varepsilon} L_G \preceq L_H \preceq e^{\varepsilon} \cdot L_G$