Announcements

- start recording
- scribe: work w/other scribe to produce one set of notes
- My OH: Mon 12:30–1:30, Thu 9–10
- TF section/OH: see Piazza
- PS 5 due Fri 12/4. Project paper drafts due Wed 12/9.
- If Zoom goes down, check Piazza
- Sync whiteboard
- For FOCS reimbursement, email Allison Choat (achoat@seas)
- Class meets Tue 12/8.

HAPPY THANKSGIVING!

Agenda

- Recap: Preconditioners (see also Peebles note on Perusall)
- Preconditioners in deterministic space $\tilde{O}(\log n)$
- Preconditioners + spectral sparsification in randomized time $\tilde{O}(m)$.

PRECONDITIONERS

$L_G x = b$

Chuzhoy et al. FOCS 2020
deterministic time $m^{1+o(1)}$

Goal: given undirected, connected G, construct symmetric Z s.t.

(1) $L_G^+ \precsim Z \precsim c \cdot L_G^+$ for "small" c

(2) Can apply Z in randomized time $\tilde{O}(m)$ / deterministic space $\tilde{O}(\log n)$

$m = \#$ edges

$\Rightarrow$ Can solve linear systems in L_G

in randomized time $\tilde{O}(m) \cdot c \log 1/\varepsilon$ / deterministic space $\tilde{O}(\log n) + O(\log n \cdot \log\log(\frac{1}{\varepsilon}))$

can be $\sqrt{c}$ using Chebyshev/CG

Via Preconditioned Richardson Iterations

(solve system $Z L_G x = Z b$, $\kappa(Z L_G) \le c$)

Last time: $Z = L_H^+$ for low-stretch spanning tree, $c = O(m \,\mathrm{polylog}\, n)$

TODAY: $c = \mathrm{polylog}(n)$

PS4: WLOG G d-regular + aperiodic; suffices to find preconditioner for $N = I - W$

idea: reduce to finding a preconditioner for $I - W^2$ + recurse $O(\log n)$ times

$$\boxed{\text{From PS4: } (I-W)^+ = \tfrac{1}{2}\left(I - J + (I+W)(I-W^2)^+(I+W)\right)}$$

Intuition (pretend invertible):

$$\left.\begin{aligned}(I-W)^{-1} &= I + W + W^2 + W^3 + \cdots \\ &= (I+W)(I+W^2)(I+W^4)(I+W^8)\cdots \\ &= (I+W)(I-W^2)^{-1}\end{aligned}\right\}$$

Above is a symmetric version of this identity

Problem with direct recursion?

— time to compute W^{2^k} = $O(k \cdot n^{\omega})$ ← matrix mult. exp. ≈ 2.37 matrices become dense

— space to compute W^{2^k} = $O(k \cdot \log n) = O(\log^2 n)$

Solution in space-bounded case: Assume G unweighted.

$W_0 = W$

W_k = derandomized square of W_{k-1} using an expander of spectral expansion $\geq 1-\omega$ and degree $c = \mathrm{poly}(1/\omega)$

Space to compute $W_k = O(\log n + k \cdot \log c)$

$$Z_{k-1} \stackrel{\text{def}}{=} \frac{1}{2}\left(I - J + (I + W_{k-1}) Z_k (I + W_{k-1})\right)$$

$$Z_{O(\log n)} \stackrel{\text{def}}{=} I - J \underset{O(\omega)}{\overset{\text{PSD}}{\approx}} I - W_{O(\log n)}$$

$x \leq (1+\varepsilon) y$

$\Downarrow$

$x + z \leq (1+\varepsilon)(y+z)$

Assume: $Z_k \underset{\delta_k}{\approx} (I - W_k)^+ \underset{O(\omega)}{\overset{\text{justify below}}{\approx}} (I - W_{k-1}^2)^+$

Then: $Z_{k-1} \underset{\delta_k + O(\omega)}{\approx} \frac{1}{2}\left(I - J + (I + W_{k-1})(I - W_{k-1}^2)^+ (I + W_{k-1})\right)$

$= (I - W_{k-1})^+$

where $A \underset{\delta}{\approx} B$ if $e^{-\delta} A \preceq B \preceq e^{\delta} A$

cf. $(1-\delta) A \preceq B \preceq (1+\delta) A$

Can set $\boxed{\delta_k = \delta_{k+1} + O(\omega)}$ for $k = 1, \dots, O(\log n)$

$\delta_{O(\log n)} = O(\omega)$

$\delta_0 = O(\omega) \cdot O(\log n)$

≤ 1 for $\omega = \frac{1}{O(\log n)}$

$Z_0 \underset{1}{\approx} (I - W)^+$

Space to apply Z_0 = space to construct $W_0, \dots, W_{O(\log n)} + O(\log n \cdot \log\log n)$

$$= O\left(\log n + O(\log n)\cdot O\left(\log \tfrac{1}{\omega}\right)\right) + O(\log n \cdot \log\log n)$$

$$= O(\log n \cdot \log\log n)$$

<u>Need to show:</u> If G_{k+1} is the derandomized square of G_k with an expander of spectral expansion $\geq 1-\omega$ for sufficiently small ω, then $I - W_{k+1} \approx_{O(\omega)} I - W_k$

<u>Proof:</u> As in PS_4, $W_{k+1} = \frac{1}{d} P (I_n \otimes W_H) P^T$ ← $L = P^T$ because G undirected

d = degree of G_k

$$\underline{I_{nd} - W_{k+1}} = \frac{1}{d} P \left(I_n \otimes \left(I_d - \underbrace{W_H}_{\approx_{O(\omega)}}\right)\right) P^T$$

$$\approx_{O(\omega)} \frac{1}{d} P \left(I_n \otimes (I_d - J_d)\right) P^T$$

$$= I_{nd} - W_k^2$$

$H_1 \approx_\varepsilon K_1$

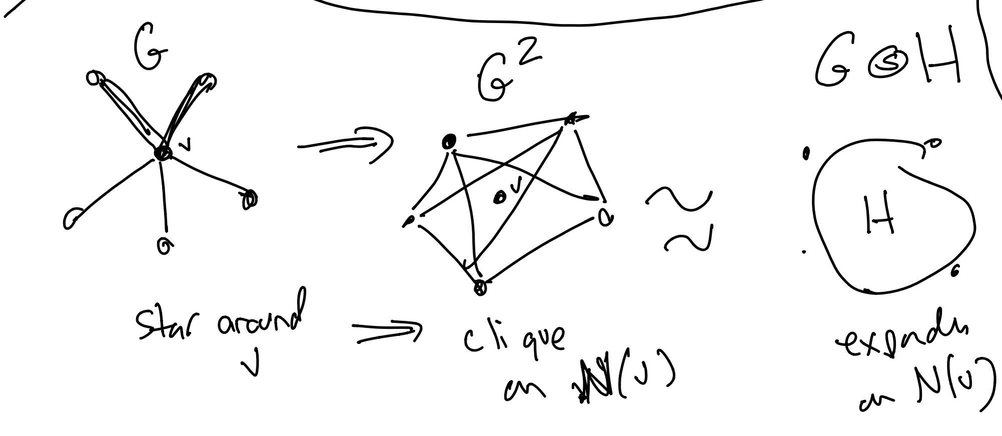
$H_2 \approx_\varepsilon K_2$

$\vdots$

$H_n \approx_\varepsilon K_n$

$\Downarrow$

$\sum H_i \approx_\varepsilon \sum K_i$ ← no loss!



Randomized Nearly Linear-Time Algorithm

Repeated derandomized squaring doesn't suffice:

$$\# \text{ edges in } W_k = m \cdot c^k \qquad k = O(\log n)$$
$$\gg \text{poly}(n)$$

Solution: spectrally sparsify at each step.

Theorem: There is a randomized algorithm that given a weighted undirected graph G w/ n vertices and m edges, $\varepsilon > 0$, outputs a weighted undirected graph H s.t. whp

1) $e^{-\varepsilon} L_G \preceq L_H \preceq e^{\varepsilon} L_G$
2) H has at most $n \cdot \text{poly}(\log n, \log(w_{max}/w_{min}), 1/\varepsilon)$ edges
3) Time to construct H is at most $O(m \cdot \text{poly}(\log n, 1/\varepsilon))$

Proof sketch (following Koutis '14)

Given $G = (V, E, w)$, we will find a set $F \subseteq E$ s.t.

(1) $|F| \leq m/4 + n \cdot \text{poly}(\log n, 1/\varepsilon_0, \log(w_{max}/w_{min}))$ edges

(2) Every edge $e = (a,b) \in E - F$ has leverage score $w_e \cdot R_{eff}(a,b) \leq \frac{\varepsilon_0^2}{c \log n}$

$\Rightarrow$ Construct $G' = (V, E', w')$ where

$$w'_e = \begin{cases} w_e & \text{if } e \in F \\ 4w_e & \text{w.p. } 1/4 \text{ if } e \in E - F \\ 0 & \text{w.p. } 3/4 \text{ if } e \in E - F \\ 0 & \text{if } e \notin E \end{cases}$$

By spectral sparsification via random sampling analysis, we have whp:

(1) $|E'| \le m/2 + n \cdot \mathrm{poly}(\log n, 1/\varepsilon_0, \log(w_{max}/w_{min}))$

(2) $e^{-\varepsilon_0} L_G \precsim L_{G'} \precsim e^{\varepsilon_0} L_G$

Recurse on G' $O(\log n)$ times to get H

with $\varepsilon = O(\log n) \cdot \varepsilon_0$

How to obtain the set F?

I. For simple graphs $G = (V, E)$:

1) Let $E_0 = E$

2) Repeat for $i = 1, \dots, t$

a) find a set R_i of at most $m/8$ edges s.t. every connected comp. of $G_i = (V, E_{i-1} - R_i)$ has diameter $\le \log_{9/8} m = O(\log m)$

b) Let F_i be a forest w/a shortest path tree in each compnent of G_i

c) Let $E_i = E_{i-1} - F_i$

3) Output $F = F_1 \cup F_2 \cup \dots \cup F_t \cup \{e \in E : e \text{ is in a majority of the } R_i\text{'s}\}$

Then: $|F| \le t \cdot (n-1) + 2 \cdot \frac{m}{8}$, and

for each $e = (a,b) \in E - F$, there are at least $\frac{t}{2}$ edge-disjoint paths of length $O(\log n)$ from a to b $\Rightarrow R_{eff}(a,b) \le \frac{O(\log n)}{t}$

$\Rightarrow$ set $t = \frac{O(\log^2 n)}{\varepsilon_0^2}$.

II. Weighted Graphs

- Bucket the edges according to weight
- $E^{(j)} = \{ e \in E : 2^j \cdot w_{min} \le w_e < 2^{j+1} \cdot w_{min} \}$ $j = 0, \ldots, k = \log_2\left(\frac{w_{max}}{w_{min}}\right)$
- Apply I to simple graph $G^{(j)} = (V, E^{(j)})$
  to obtain set $F^{(j)} \subseteq B^{(j)}$
- Output $F = F^{(0)} \cup \cdots \cup F^{(k)}$

. $$|F| \le \sum_{j=0}^{k} \left(\frac{|F^{(j)}|}{4} + n \cdot \text{poly}(\log n, 1/\varepsilon) \right)$$
$$= \frac{m}{4} + n \cdot \text{poly}(\log n, 1/\varepsilon) \cdot \log\left(\frac{w_{max}}{w_{min}}\right)$$

. For each edge $e = (a,b) \in E - F$, $e \in E^{(j)} - F^{(j)}$ for some j, so
$$w_e \cdot R_{\text{eff}}^{G}(a,b) \le w_e \cdot \frac{R_{\text{eff}}^{G^{(j)}}(a,b)}{w_{min} \cdot 2^j} \le 2 \cdot R_{\text{eff}}^{G^{(j)}}(a,b) \le \frac{O(\log n)}{t}$$