Announcements

- start recording
- scribe: work w/other scribe to produce one set of notes
- My OH: Mon 12:30–1:30, Thu 9–10
- TF section/OH: Mon 2–3, Tue 8–9, Wed 5–6, Thu 5:30–6:30
- PS 4 posted. Project proposal due Sun 11/8, problems due TUE 11/17
- If Zoom goes down, check Piazza
- Sync whiteboard
- Post your project topic on spreadsheet
- Last project check-ins: today 3–3:45, Wed 4–5, Thu 9–10

Agenda

- Recap: spectral sparsification
- Reductions between algorithmic problems
- Iterative methods

Recap: Spectral Sparsification

Thm: $\forall$ weighted undirected G on n vertices $\exists$ weighted undirected H w/ $O\left(\frac{n \log n}{\varepsilon^2}\right)$ edges

such that $(1-\varepsilon) L_G \preceq L_H \preceq (1+\varepsilon) L_G$

Proof via Randomized Algorithm:

Include each edge e of G in H w.p. p_e, independently for each e

$$w_H(e) = \begin{cases} \frac{w_G(e)}{p_e} & \text{w.p. } p_e \\ 0 & \text{w.p. } 1-p_e \end{cases}$$

$$p_e = c \cdot \ell_e = c \cdot w_G(e) \cdot R_{eff}(a,b) \quad \text{for } e=(a,b)$$

for $c = O\left(\frac{\log n}{\varepsilon^2}\right)$ $\qquad R_{eff}(a,b) = (\delta_a - \delta_b)^T L^+ (\delta_a - \delta_b)$

- $E\left[\# \text{ edges in } H\right] = \sum_e p_e \leq c \cdot n$
- $\Pr\left[L_H \text{ does not } \varepsilon\text{-approx } L_G\right] \leq 2n \cdot e^{-\varepsilon^2/3R} = 2n \cdot e^{-c\varepsilon^2/3}$

$$R = \max_{e=(a,b)} \left\| \frac{w_G(e)}{p_e} \cdot L_G^{+/2} L_{(a,b)} L_G^{+/2} \right\|$$

$$= \max_{e=(a,b)} \frac{\ell_e}{p_e}$$

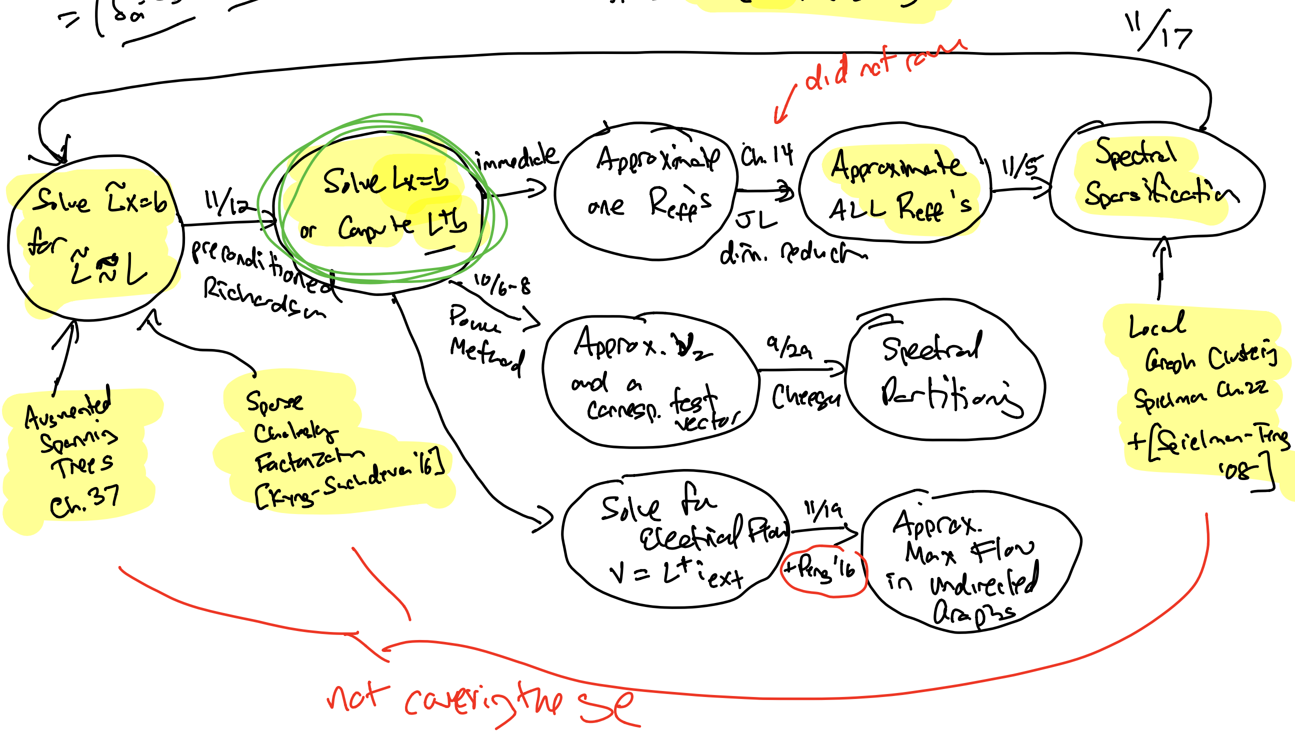
$\Rightarrow$ suffices to set p_e using constant-factor approximations of ℓ_e (or $R_{eff}(a,b)$)

(e.g. $\hat{\ell}_e \in \left[\frac{\ell_e}{2}, 2\ell_e\right]$)

Reductions between randomized nearly linear-time algorithms

time $O(m \cdot \text{polylog } n)$

$R_{eff}(a,b)$
$= \|L^{+/2}\delta_a - L^{+/2}\delta_b\|^2$
$= (\delta_a - \delta_b)^T L^+ (\delta_a - \delta_b)$



Richardson Iterations

Goal: solve $Ax=b$ (e.g. $A=$ a Laplacian or normalized Laplacian restricted to $v_1^\perp$)

without Gaussian elimination (only applying A to vectors)

$\alpha=1$ $A=I-W$ } for expanders

$I-\alpha A=W$

Algorithm: $x^{(0)}=\vec{0}$

$$x^{(t+1)}=(I-\alpha A)x^{(t)}+\alpha b$$

$$=x^{(t)}-\alpha(Ax^{(t)}-b)$$

when A is symmetric gradient descent on obj function $f(x)=\frac{1}{2}x^TAx-b^Tx$

Analysis: Suppose $Ax=b$

Then $x^{(t+1)}-x=(I-\alpha A)(x^{(t)}-x)$

$\|M\|=\max_{y\neq 0}\frac{\|My\|}{\|y\|}$ $\Rightarrow x^{(t)}\to x$ if $\|I-\alpha A\|<1$

$$\|x^{(t+1)}-x\|\leq\|I-\alpha A\|^{t+1}\cdot\|x\|$$

$t=O\left(\frac{\log(\frac{1}{\varepsilon})}{1-\|I-\alpha A\|}\right)$ iterations suffices to get $\|x^t-x\|\leq\varepsilon\cdot\|x\|$

holds even for asymmetric A

When A symmetric, $\|I-\alpha A\|=\max\{|1-\alpha\lambda_1|,|1-\alpha\lambda_n|\}$

$\lambda_1\leq\cdots\leq\lambda_n$ e-values of A

Optimizing α: $(1-\alpha\lambda_1)=-(1-\alpha\lambda_n)$

$$\alpha=\frac{2}{\lambda_1+\lambda_n}\qquad\|I-\alpha A\|=\frac{\lambda_n-\lambda_1}{\lambda_1+\lambda_n}$$

time to solve $Lx=b$ is $O(n\cdot\log(\frac{1}{\varepsilon}))$

$\#$ iterations $=O\left(\left(1+\frac{\lambda_n}{\lambda_1}\right)\cdot\log\left(\frac{1}{\varepsilon}\right)\right)$ also measures numerical stability

$\kappa(A)$ "condition number"

on expanders $\kappa(L_G^{1^\perp})=\frac{\lambda_n}{\lambda_2}=O(1)$

Similarly $\underbrace{Ax^{(t+1)} - b}_{\text{"residual"}} = \underline{(I - \alpha A)(Ax^{(t)} - b)}$

$\Rightarrow$ Same # iterations gives $\boxed{\|Ax^{(t)} - b\| \leq \varepsilon \cdot \|b\|.}$

Another view: $x^{(t)} = p_t(A)b$ for a polynomial p_t

Richardson: $p_t(A) = \alpha \cdot \sum_{j=0}^{t} (I - \alpha A)^j \longrightarrow A^{-1}$ as $t \to \infty$

$\sum_{j=0}^{\infty} x^j = (1-x)^{-1}$

for $t = t(\kappa, \varepsilon)$

$$\begin{cases} \forall x \quad \| \overbrace{p_t(A)(Ax)}^{x^{(t)}} - x \| \leq \varepsilon \cdot \|x\| \qquad = (Ap_t(A) - I)x \\ \text{i.e.} \quad \underbrace{\| A\, p_t(A) - I \| \leq \varepsilon}_{\|AZ - I\| \leq \varepsilon} \end{cases}$$

Richardson: suffices to take degree $t = O\left(\frac{\lambda_n}{\lambda_1} \cdot \log\left(\frac{1}{\varepsilon}\right)\right)$

Chebychev polys: degree $O\left(\sqrt{\frac{\lambda_n}{\lambda_1}} \cdot \log\left(\frac{1}{\varepsilon}\right)\right)$

best possible.

even undirected n-cycle $\quad d=2$

$$\frac{\lambda_n}{\lambda_2} = \frac{2 \cdot d}{\Theta\left(\frac{1}{n^2}\right)} = \Theta(n^2)$$

$\rightarrow$ time $O\left(n \cdot m \cdot \log\left(\frac{1}{\varepsilon}\right)\right)$

Another notion of solving: spectral approximation of A^{-1}
for positive definite A

$$(1-\varepsilon)A^{-1} \preceq Z \preceq (1+\varepsilon)A^{-1}$$

Prop: Z ε-spectral approx of A^{-1}

$$\iff \|ZA - I\|_A \le \varepsilon$$

where $\|x\|_A \overset{\text{def}}{=} \sqrt{x^T \underline{A} x}$

$$\|M\|_A = \max_{x \neq 0} \frac{\|Mx\|_A}{\|x\|_A}$$

Proof:

Note: if Z, A commute, then $\|ZA - I\|_A =$