Announcements

- include pronouns in Zoom name - mute self when no
- videos on, don't record other students or share class recordings
- TFs: Chi-Ning Chou, Alec Sun, Santhoshini Velusamy, + TBD
- scribe
- start recording
- recordings of 8/20 overview, 9/2 section on Piazza
- PS0 due Fri 9/11
- stay tuned for poll re section + more TF OH
- Mon 9/7 holiday (no OH for me)
- use Piazza to find pset partners
- in case Zoom goes down, check Piazza

Graphs

most general for us = weighted digraph

$G=(V,E)$ $E \subseteq V \times V$ $w: E \to \mathbb{R}^+$

allow self-loops (v,v)

Equivalently

$w: V \times V \to \mathbb{R}^{\geq 0}$

$E = \{ e \in V \times V : w(e) > 0 \}$

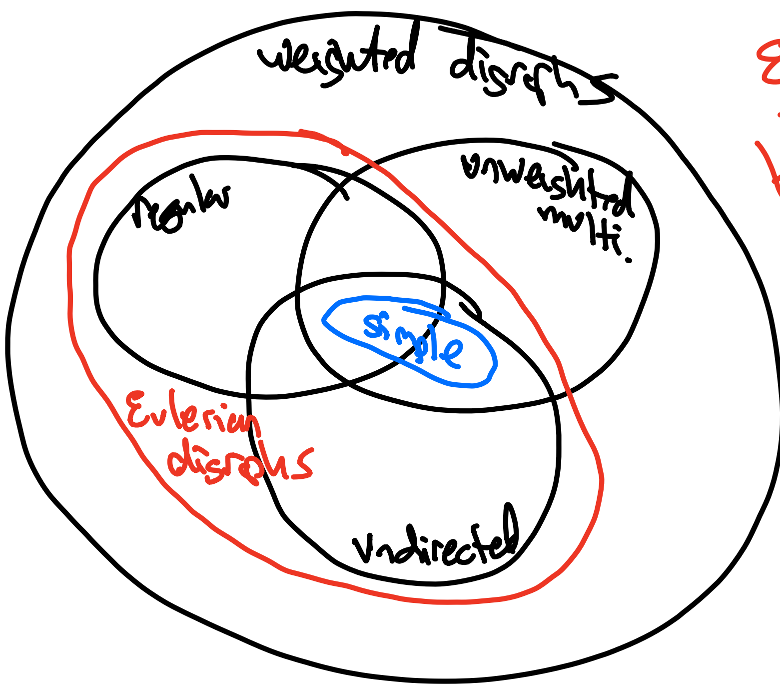
Special cases

- undirected: $w(a,b) = w(b,a) \quad \forall a,b$
- unweighted multigraph: $w : V \times V \to \mathbb{N}$

  $w(a,b)$ = multiplicity of (a,b)
- regular: $\exists d \ \forall a \quad d_{out}(a) = d_{in}(a) = d$

$$d_{out}(a) \overset{def}{=} \sum_b w(a,b)$$

$$d_{in}(a) = \sum_b w(b,a)$$



Eulerian digraphs

$\forall a \quad d_{in}(a) = d_{out}(a)$

Simple graph

undirected

unweighted

no parallel edges

no self-loop

Matrices

M_G

- adjacency matrix $M(a, \underline{b}) = \cancel{w(a,b)}\ w(b,a)$
- random-~~walk~~ matrix $W = M \cdot D_{out}^{-1}$
  diffusin

$$D_{out} = \begin{pmatrix} d_{out}(1) & & 0 \\ 0 & \ddots & \\ & & d_{out}(n) \end{pmatrix}$$

$$W(a,b) = \frac{w(b,a)}{d_{out}(b)}$$

$$D_{in} = \dots$$

In Eulerin digraph$^{(*)}$, $D_{in} = D_{out} = D$

(*) Laplacian: $L = D - M$

RW Laplacin: $L_{rw} = I - W \overset{(*)}{=} \underline{L D^{-1}}$

(*) Normalized Laplacian: $N = I - D^{-1/2} M D^{-1/2}$

$$= D^{-1/2} L D^{-1/2}$$

Assume G Eulerian! $\quad = D^{-1/2} L_{rw} D^{1/2}$

Matrix	e-vector	e-value	symmetric if G undirected?
L	$\vec{1}$	0	✓
$L_{rw} = I - W$	$\vec{d} = D\vec{1}$	0	not nec. when G irregular
W	$\vec{d}$	1	"
N	$\vec{d}^{1/2}$	0	✓
A	??		✓

$$W\vec{d}_{out} = \vec{d}_{in}$$

Note: when G is regular all of these are equivalent to study

$$D = dI$$

Spectral Thm: If M is a symmetric matrix, then $M = V \Lambda V^T$ for an orthogonal matrix V $\left(\begin{array}{c} VV^T = I \\ V^TV = I \\ \text{cols/rows orthonormal} \end{array} \right)$ and a diagonal matrix Λ.

$$M = \sum_{i=1}^{n} \lambda_i v_i v_i^T$$

λ_i: diag of Λ — v_i: cols of V

equivalently $v_1, \dots, v_n$ are an orthonormal basis of e-vectors of M w/ e-values $\lambda_1, \dots, \lambda_n$

For asymmetric matrices, we have SVD $M = U \Sigma V^T$ for orthogonal U, V, nonneg diag Σ

Pf outline:

① Let $\lambda_1 = \max_{\|x\|=1} \underbrace{\frac{x^T M x}{x^T x}}$ } Rayleigh quotient

and v_1 = corresponding unit vector x

② Show that v_1 is an evector of eigenvalue λ_1

(pf: calculus - gradient $= 0$)

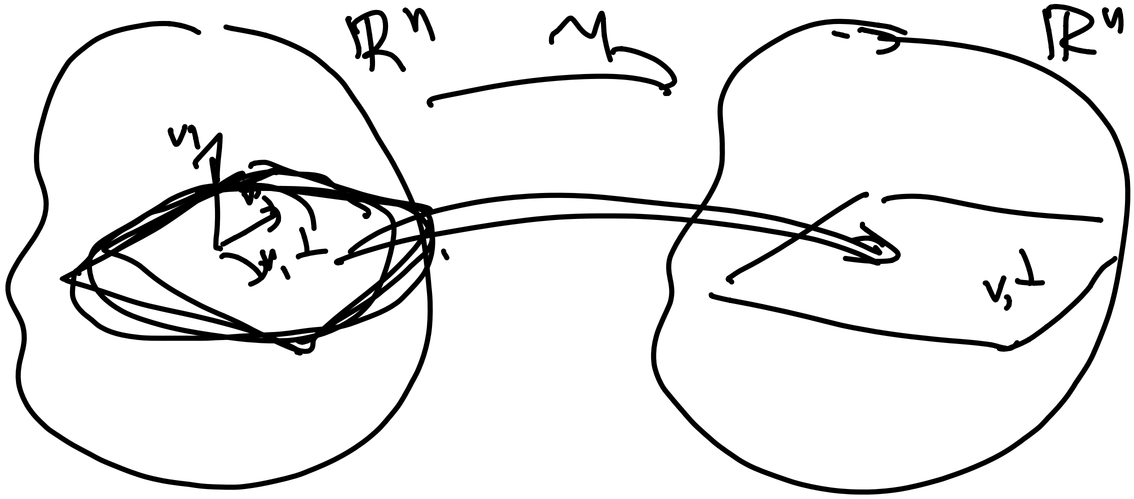
③ Restrict to $(n-1)$-dimensional subspace

$$v_1^\perp = \{w : w^T v_1 = 0\}$$

check: $M v_1^\perp \subseteq v_1^\perp$

$(v_1^T M w = \lambda_1 v_1^T w = 0)$

④ Apply induction on dimension to $M\big|_{v_1^\perp}$ = M restricted to $v_1^\perp$



- Get orthonormal basis of e-vectors on $v_1^\perp$
- Add v_1 to get basis of $\mathbb{R}^n$.