1 Recap / Correction from Last Time

Complete graphs without self-loops

1. The Laplacian matrix of a complete graph has eigenvalues $\lambda_2 = \lambda_3 = \cdots = \lambda_n = n = d + 1$
2. Largest possible value of $\frac{\lambda_2}{d} = \frac{n}{n - 1}$.

Proof. Let $L$ be the Laplacian of any $d$–regular graph.

$$d \cdot n = \text{Tr}(L) = \sum_{i=1}^{n} \lambda_i \geq (n - 1)\lambda_2$$

$$\Rightarrow \frac{n}{n - 1} \geq \frac{\lambda_2}{d}$$

This means that asymptotically, $\frac{\lambda_2}{d} \to 1$, but it can take a higher value (it is 2 for $n = 2$).

3. Largest possible value of $\frac{\lambda_n}{d} = 2$ (we will see this in more detail in a later class; it is achieved in any connected bipartite graph).

2 Diagonalization on $\mathbb{C}$

Theorem 1 (Spectral Theorem on $\mathbb{R}$). Let $M \in \mathbb{R}^{n \times n}$ ($n \times n$ real matrices), then the following are equivalent (TFAE):

1. There exists an orthonormal basis $v_1, \ldots, v_n \in \mathbb{R}^n$ of real eigenvectors of $M$
2. $M = V\Lambda V^\top$ for orthogonal matrix $V \in \mathbb{R}^{n \times n}$ and diagonal $\Lambda \in \mathbb{R}^{n \times n}$
3. $M$ is symmetric

But what are the complex analogues of the above objects?

- For $z = x + iy \in \mathbb{C}$, $z^* = x - iy$
- For $v \in \mathbb{C}^n, M \in \mathbb{C}^{n \times n}, v^*, M^*$ are called conjugate transposes
- $\|v\| = \sqrt{v^*v}$ (norm), $\langle v, w \rangle = v^*w$ (inner product)
- Orthonormal basis of $\mathbb{C}^{n \times n}$: $v_1, \ldots, v_n \in \mathbb{C}^n$ s.t. $\langle v_i, v_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$
- Unitary matrix $V \in \mathbb{C}^{n \times n}$. $V^*V = I$
- $M$ is Hermitian if $M^* = M$
Theorem 2. For $M \in \mathbb{C}^{n \times n}$, the following are equivalent:

1. There exists orthonormal basis $v_1, \ldots, v_n \in \mathbb{C}^n$ of complex eigenvectors of $M$
2. $M = V^* \Lambda V$ for unitary $V$ and diagonal $\Lambda$
3. $M$ is normal: $M^* M = M M^*$. Note: This is a more general condition than $M$ being Hermitian or symmetric (even for real matrices).

3 Groups

Definition 3 (Group). A group is a set $\Gamma$ with a binary operation $\circ$ such that:

1. $\forall x, y, z \in \Gamma$, $(x \circ y) \circ z = x \circ (y \circ z)$ (associativity)
2. $\exists e \in \Gamma$ s.t. $\forall x \in \Gamma$, $e \circ x = x \circ e = x$ (identity)
3. $\forall x \in \Gamma$, $\exists y \in \Gamma$, $x \circ y = y \circ x = e$ (inverses)

Definition 4 (Abelian Group). A group $(\Gamma, \circ)$ is abelian if $\forall x, y \in \Gamma$: $x \circ y = y \circ x$ (commutativity).

Examples of groups

1. $(\mathbb{R}, +)$ - addition on the set of real numbers
2. $(\mathbb{R}^{n \times n}, +)$ - addition of the set of real, square matrices
3. $(\mathbb{Z}, +)$ - addition of the set of integers
4. $\mathbb{Z}_n = (\{0, \ldots, n-1\}, + \mod n) \cong \mathbb{Z}/n\mathbb{Z}$ - integers ”modulo” equivalence relation ( $a \equiv b$ if $n|a-b$ ; ‘$\equiv$’: ‘congruent to’)
5. $(\{0,1\}^d, \text{bitwise} \oplus) \cong (\mathbb{Z}/n\mathbb{Z})^d$ - bitwise xor on the $d$-dimensional hypercube
6. Every finite abelian group is $\cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$
7. $\mathbb{R}^* = (\mathbb{R} - \{0\}, \times)$, $\mathbb{C}^* = (\mathbb{C} - \{0\}, \times)$
8. $S^1 = (\{z \in \mathbb{C} : |z| = 1\}, \times)$ (where inverse of $z = z^*$)
9. $(n \times n$ invertible real matrices, $\times$) - nonabelian when $n > 1$
10. $(n \times n$ unitary complex matrices, $\times$) - nonabelian when $n > 1$

4 General Cayley Digraphs

Definition 5 (Cayley Digraph). For a finite group $(\Gamma, +)$ and a subset $S \subseteq \Gamma$, $\text{Cay}(\Gamma, S)$ is the $|S|$-regular digraph with

- vertex set: $\Gamma$
- edges: $\{(x, x+s) : s \in S\}$

It is connected iff $S$ “generates” $\Gamma$.  

CS 229r Spectral Graph Theory in Computer Science, Lecture 3-2
Examples

1. Hypercube: \( \Gamma = \{0, 1\}^d, S = \{e_1, \ldots, e_d\} \)
2. Directed \( n \)-cycle: \( \Gamma = \mathbb{Z}_n \) (which is abelian), \( S = \{1\} \)
3. Undirected \( n \)-cycle: \( \Gamma = \mathbb{Z}_n, S = \{1, -1\} \equiv \{1, n - 1\} \)
4. Complete graph (with self-loops): \( S = \Gamma \)

Weighted Cayley Digraphs

Definition 6 (Weighted Cayley Digraphs). Given a weight function on the group, \( w_0 : \Gamma \to \mathbb{R}^\geq 0 \), \( \text{Cay}(\Gamma, w_0) \) has:

- **vertex set**: \( \Gamma \)
- **edge weights**: \( w(a, b) = w_0(b - a) \)
- **\( d \)-regular** with \( d = \sum_{s \in \Gamma} w_0(s) \)

Note that this definition is more general than the previous one. More specifically, \( \text{Cay}(\Gamma, S) = \text{Cay}(\Gamma, 1_S) \), where \( 1_S \) is the indicator for set \( S \).

Example: Noisy Hypercube, \( NH_p \) for \( p \in [0, 1] \)

For \( s \in \{0, 1\}^d \):

\[
w(s) = p^{|s|} \cdot (1 - p)^{d - |s|} = \Pr(s) = \text{Bern}(p)^d
\]

where \( |s| \) is the number of 1’s in \( s \) (which is the ‘Hamming weight’ of \( s \)).

Finite abelian groups

Let \( \Gamma \) be a finite abelian group, \( S \subseteq \Gamma \), then:

1. \( M \) is the adjacency matrix of \( \text{Cay}(\Gamma, S) \)
2. \( W = M/d \) is the random walk matrix, where \( d = |S| \)
3. \( L = I - W \) is the normalized Laplacian

Claim 7. \( M \) (and hence \( W, L \)) is normal

Proof. We note that the \( M \) is a real matrix, and \( M^* M, MM^* \) have nice interpretations themselves (in terms of Cayley graphs):

\[
M^* M = M^T M = \text{Cay}(\Gamma, \{s - t : s, t \in S\})
\]

\[
MM^* = MM^T = \text{Cay}(\Gamma, \{-s + t : s, t \in S\})
\]

where the sets \( \{s - t : s, t \in S\}, \{-s + t : s, t \in S\} \) are with multiplicity. Because the group is abelian, we will have the two sets being the same.

In general, for every \( S, T \), \( \text{Cay}(\Gamma, S) \) and \( \text{Cay}(\Gamma, T) \) commute with each other (i.e. their adjacency matrices commute). This further implies that the adjacency matrices have a common diagonalization, with a common basis for all choices of \( S, T \). The Fourier eigenbasis is that basis, and lets us get an explicit handle on what that basis is.
5 Fourier eigenbasis

Theorem 8. Assume $\Gamma$ to be abelian. An orthogonal set of complex eigenvectors for $\text{Cay}(\Gamma, S)$ (for $M$, $L$, and $W$) is given by the set of characters $\chi : \Gamma \to \mathbb{C}$, where:

$$\chi : \Gamma \to S' \subseteq \mathbb{C}$$

s.t. $\forall x, y$: $\chi(x + y) = \chi(x)\chi(y)$ (homomorphism)

Examples

1. Characters for $\mathbb{Z}_n = \{0, \ldots, n-1\}$, $\mod n$: $\chi_r(x) = e^{2\pi irx/n}$, $r \in \{0, \ldots, n-1\}$.

For $n = 4$:

<table>
<thead>
<tr>
<th>$r$</th>
<th>$\chi(0)$</th>
<th>$\chi(1)$</th>
<th>$\chi(2)$</th>
<th>$\chi(3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>$i$</td>
<td>-1</td>
<td>-$i$</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>-$i$</td>
<td>-1</td>
<td>$i$</td>
</tr>
</tbody>
</table>

2. Characters for hypercube $\cong \mathbb{Z}_2^d$.

For $r \in \{0,1\}^d$:

$$\chi_r(x) = (-1)^{r_1x_1}(-1)^{r_2x_2} \ldots (-1)^{r_nx_n} = (-1)^{\langle r, x \rangle}$$

6 Calculating eigenvalues

$$(M\chi)(a) = \sum_{b \in \Gamma} M(a, b)\chi(b)$$

$$= \sum_{b \in \Gamma} w(b, a)\chi(b)$$

$$= \sum_{b \in \Gamma} w_0(a - b)\chi(b)$$

$$= \sum_{s \in \Gamma} w_0(s)\chi(a - s)$$

$$= \sum_{s \in \Gamma} w_0(s)\chi(a)\chi(-s)$$ (homomorphism)

$$= \chi(a)\sum_{s \in \Gamma} w_0(s)\chi(s)^* \quad \left(= \chi(a)\sum_{s \in \Gamma} \chi(s)^* \text{ for unweighted case} \right)$$

$$= \chi(a)\lambda$$

Here we find that the eigenvalue corresponding to the eigenvector $\chi$ is $\sum_{s \in \Gamma} w_0(s)\chi(s)^*$

CS 229r Spectral Graph Theory in Computer Science, Lecture 3-4
7 Exercise for the next class

1. Work out the $n$ eigenvalues of directed $n$–cycle (Trevisan does undirected $n$–cycles)

2. Work out the $2^d$ eigenvalues of the Noisy Hypercube, $NH_p$ (Trevisan does ordinary hypercube)