Unifying computational entropies via Kullback–Leibler divergence

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Abstract

We introduce KL-*hardness*, a new notion of hardness for search problems which on the one hand is satisfied by all one-way functions and on the other hand implies both *next-block pseudoentropy* and *inaccessible-entropy*, two forms of computational entropy used in recent constructions of pseudorandom generators and statistically hiding commitment schemes, respectively. Thus, KL-hardness unifies the latter two notions of computational entropy and sheds light on the apparent "duality" between them. Additionally, it yields a more modular and illuminating proof that one-way functions imply next-block inaccessible entropy, similar in structure to the proof that one-way functions imply next-block pseudoentropy (Vadhan and Zheng, STOC '12).

Keywords: one-way function, pseudorandom generator, pseudoentropy, computational entropy, inaccessible entropy, statistically hiding commitment, next-bit pseudoentropy.

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1 Introduction

1.1 One-way functions and computational entropy

One-way functions [DH76] are on one hand the minimal assumption for complexity-based cryptography [IL89], but on the other hand can be used to construct a remarkable array of cryptographic primitives, including such powerful objects as CCA-secure symmetric encryption, zero-knowledge proofs and statistical zero-knowledge arguments for all of **NP**, and secure multiparty computation with an honest majority [GGM86, GMW91, GMW87, HILL99, Rom90, Nao91, HNO+09]. All of these constructions begin by converting the "raw hardness" of a one-way function (OWF) to one of the following more structured cryptographic primitives: a pseudorandom generator (PRG) [BM82, Yao82], a universal one-way hash function (UOWHF) [NY89], or a statistically hiding commitment scheme (SHC) [BCC88].

The original constructions of these three primitives from arbitrary one-way functions [HILL99, Rom90, HNO+09] were all very complicated and inefficient. Over the past decade, there has been a series of simplifications and efficiency improvements to these constructions [HRVW09, HRV13, HHR+10, VZ12], leading to a situation where the constructions of two of these primitives — PRGs and SHCs — share a very similar structure and seem "dual" to each other. Specifically, these constructions proceed as follows:

1. Show that every OWF $f : \{0,1\}^n \to \{0,1\}^n$ has a gap between its "real entropy" and an appropriate form of "computational entropy". Specifically, for constructing PRGs, it is shown that the function $\mathsf{G}(x) = (f(x), x_1, x_2, \ldots, x_n)$ has "next-block pseudoentropy" at least $n + \omega(\log n)$ while its real entropy is $\mathrm{H}(\mathsf{G}(U_n)) = n$ [VZ12] where $\mathrm{H}(\cdot)$ denotes Shannon entropy. For constructing SHCs, it is shown that the function $\mathsf{G}(x) = (f(x)_1, \ldots, f(x)_n, x)$ has "next-block accessible entropy" at most $n - \omega(\log n)$ while its real entropy is again $\mathrm{H}(\mathsf{G}(U_n)) = n$ [HRVW09]. Note that the differences between the two cases are whether we break x or $f(x)$ into individual bits (which matters because the "next-block" notions of computational entropy depend on the block structure) and whether the form of computational entropy is larger or smaller than the real entropy.

2. An "entropy equalization" step that converts G into a similar generator where the real entropy in each block conditioned on the prefix before it is known. This step is exactly the same in both constructions.

3. A "flattening" step that converts the (real and computational) Shannon entropy guarantees of the generator into ones on (smoothed) min-entropy and max-entropy. This step is again exactly the same in both constructions.

4. A "hashing" step where high (real or computational) min-entropy is converted to uniform (pseudo)randomness and low (real or computational) max-entropy is converted to a small-support or disjointness property. For PRGs, this step only requires randomness extractors [HILL99, NZ96], while for SHCs it requires (information-theoretic) interactive hashing [NOVY98, DHRS04]. (Constructing full-fledged SHCs in this step also utilizes UOWHFs, which can be constructed from one-way functions [Rom90]. Without UOWFHs, we obtain a weaker binding property, which nevertheless suffices for constructing statistical zero-knowledge arguments for all of **NP**.)

This common construction template came about through a back-and-forth exchange of ideas between the two lines of work. Indeed, the uses of computational entropy notions, flattening, and hashing originate with PRGs [HILL99], whereas the ideas of using next-block notions, obtaining them from breaking $(f(x), x)$ into short blocks, and entropy equalization originate with SHCs [HRVW09]. All this leads to a feeling that the two constructions, and their underlying computational entropy notions, are "dual" to each other and should be connected at a formal level.

In this paper, we make progress on this project of unifying the notions of computational entropy, by introducing a new computational entropy notion that yields both next-block pseudoentropy and next-block accessible entropy in a clean and modular fashion. It is inspired by the proof of [VZ12] that $(f(x), x_1, \ldots, x_n)$ has next-block pseudoentropy $n + \omega(\log n)$, which we will describe now.

1.2 Next-block pseudoentropy via KL-hardness

We recall the definition of next-block pseudoentropy, and the result of [VZ12] relating it to one-wayness.

Definition 1.1 (next-block pseudoentropy, informal)**.** *Let n be a security parameter, and $X = (X_1, \ldots, X_m)$ be a random variable distributed on strings of length* poly(n)*. We say that X has* next-block pseudoentropy *at least k if there is a random variable $Z = (Z_1, \ldots, Z_m)$, jointly distributed with X, such that:*

1. *For all $i = 1, \ldots, m$, $(X_1, \ldots, X_{i-1}, X_i)$ is computationally indistinguishable from $(X_1, \ldots, X_{i-1}, Z_i)$.*
2. $\sum_{i=1}^{m} \mathrm{H}(Z_i | X_1, \ldots, X_{i-1}) \geq k$.

Equivalently, for I uniformly distributed in $[m]$, X_I has conditional pseudoentropy *at least k/m given $(X_1, \ldots, X_{i-1})$.*

It was conjectured in [HRV10] that next-block pseudoentropy could be obtained from any OWF by breaking its input into bits, and this conjecture was proven in [VZ12]:

Theorem 1.2 ([VZ12], informal)**.** *Let $f : \{0,1\}^n \to \{0,1\}^n$ be a one-way function, let X be uniformly distributed in $\{0,1\}^n$, and let $X = (X_1, \ldots, X_m)$ be a partition of X into blocks of length $O(\log n)$. Then $(f(X), X_1, \ldots, X_m)$ has next-block pseudoentropy at least $n + \omega(\log n)$.*

The intuition behind Theorem 1.2 is that since X is hard to sample given $f(X)$, then it should have some extra computational entropy given $f(X)$. This intuition is formalized using the following notion of being "hard to sample":

Definition 1.3 (KL-hard for Sampling)**.** *Let n be a security parameter, and (X, Y) be a pair of random variables, jointly distributed over strings of length* poly(n)*. We say that X is* Δ-KL-hard for sampling *given Y if for all probabilistic polynomial-time S, we have*

$$\mathrm{KL}\left(X, Y \,\|\, \mathsf{S}(Y), Y\right) \geq \Delta,$$

where $\mathrm{KL}(\cdot \,\|\, \cdot)$ denotes Kullback–Leibler divergence (a.k.a. relative entropy).[1]

[1] Recall that for random variables A and B with $\mathrm{Supp}(A) \subseteq \mathrm{Supp}(B)$, the Kullback–Leibler divergence is defined by $\mathrm{KL}(A \,\|\, B) = \mathrm{E}_{a \leftarrow A}\left[\log(\Pr[A = a] / \Pr[B = a])\right]$.

That is, it is hard for any efficient adversary S to sample the conditional distribution of X given Y, even approximately.

The first step of the proof of Theorem 1.2 is to show that one-wayness implies KL-hardness of sampling (which can be done with a one-line calculation):

Lemma 1.4. *Let $f : \{0,1\}^n \to \{0,1\}^n$ be a one-way function and let X be uniformly distributed in $\{0,1\}^n$. Then X is $\omega(\log n)$-KL-hard to sample given $f(X)$.*

Next, we break X into short blocks, and show that the KL-hardness is preserved:

Lemma 1.5. *Let n be a security parameter, let (X, Y) be random variables distributed on strings of length $\mathrm{poly}(n)$, let $X = (X_1, \ldots, X_m)$ be a partition of X into blocks, and let I be uniformly distributed in $[m]$. If X is Δ-KL-hard to sample given Y, then X_I is (Δ/m)-KL-hard to sample given $(Y, X_1, \ldots, X_{I-1})$.*

Finally, the main part of the proof is to show that, once we have short blocks, KL-hardness of sampling is *equivalent* to a gap between conditional pseudoentropy and real conditional entropy.

Lemma 1.6. *Let n be a security parameter, Y be a random variable distributed on strings of length $\mathrm{poly}(n)$, and X a random variable distributed on strings of length $O(\log n)$. Then X is Δ-KL-hard to sample given Y iff X has conditional pseudoentropy at least $\mathrm{H}(X|Y)+\Delta$ given Y.*

Putting these three lemmas together, we see that when f is a one-way function, and we break X into blocks of length $O(\log n)$ to obtain $(f(X), X_1, \ldots, X_m)$, on average, the conditional pseudoentropy of X_I given $(f(X), X_1, \ldots, X_{I-1})$ is larger than its real conditional entropy by $\omega(\log n)/m$. This tells us that the next-block pseudoentropy of $(f(X), X_1, \ldots, X_m)$ is larger than its real entropy by $\omega(\log n)$, as claimed in Theorem 1.2.

We remark that Lemma 1.6 explains why we need to break the input of the one-way function into short blocks: it is false when X is long. Indeed, if f is a one-way function, then we have already seen that X is $\omega(\log n)$-KL-hard to sample given $f(X)$ (Lemma 1.4), but it does not have conditional pseudoentropy noticeably larger than $\mathrm{H}(X|f(X))$ given $f(X)$ (as correct preimages can be efficiently distinguished from incorrect ones using f).

1.3 Inaccessible entropy

As mentioned above, for constructing SHCs from one-way functions, the notion of next-block pseudoentropy is replaced with next-block accessible entropy:

Definition 1.7 (next-block inaccessible entropy, informal)**.** *Let n be a security parameter, and $Y = (Y_1, \ldots, Y_m)$ be a random variable distributed on strings of length $\mathrm{poly}(n)$. We say that Y has* next-block accessible entropy *at most k if the following holds.*

Let $\widetilde{\mathsf{G}}$ be any probabilistic $\mathrm{poly}(n)$-time algorithm that takes a sequence of uniformly random strings $\widetilde{R} = (\widetilde{R}_1, \ldots, \widetilde{R}_m)$ and outputs a sequence $\widetilde{Y} = (\widetilde{Y}_1, \ldots, \widetilde{Y}_m)$ in an "online fashion" by which we mean that $\widetilde{Y}_i = \widetilde{\mathsf{G}}(\widetilde{R}_1, \ldots, \widetilde{R}_i)$ depends on only the first i random strings of $\widetilde{\mathsf{G}}$ for $i = 1, \ldots, m$. Suppose further that $\mathrm{Supp}(\widetilde{Y}) \subseteq \mathrm{Supp}(Y)$.

Then we require:

$$\sum_{i=1}^{m} \mathrm{H}\left(\widetilde{Y}_i | \widetilde{R}_1, \ldots, \widetilde{R}_{i-1}\right) \leq k.$$

(Next-block) accessible entropy differs from (next-block) pseudoentropy in two ways:

1. Accessible entropy is useful as an *upper* bound on computational entropy, and is interesting when it is *smaller* than the real entropy $\mathrm{H}(Y)$. We refer to the gap $\mathrm{H}(Y) - k$ as the *inaccessible entropy* of Y.

2. The accessible entropy adversary $\widetilde{\mathsf{G}}$ is trying to *generate* the random variables Y_i conditioned on the history rather than recognize them. Note that we take the "history" to not only be the previous blocks $(\widetilde{Y}_1, \ldots, \widetilde{Y}_{i-1})$, but the coin tosses $(\widetilde{R}_1, \ldots, \widetilde{R}_{i-1})$ used to generate those blocks.

Note that one unsatisfactory aspect of the definition is that when the random variable Y is not *flat* (i.e. uniform on its support), then there can be an adversary $\widetilde{\mathsf{G}}$ achieving accessible entropy even *larger* than $\mathrm{H}(Y)$, for example by making $\widetilde{Y}$ uniform on $\mathrm{Supp}(Y)$.

Similarly to (and predating) Theorem 1.2, it is known that one-wayness implies next-block inaccessible entropy.

Theorem 1.8 ([HRVW09]). *Let $f : \{0,1\}^n \to \{0,1\}^n$ be a one-way function, let X be uniformly distributed in $\{0,1\}^n$, and let $(Y_1, \ldots, Y_m)$ be a partition of $Y = f(X)$ into blocks of length $O(\log n)$. Then $(Y_1, \ldots, Y_m, X)$ has next-block accessible entropy at most $n - \omega(\log n)$.*

Unfortunately, however, the existing proof of Theorem 1.8 is not modular like that of Theorem 1.2. In particular, it does not isolate the step of relating one-wayness to entropy-theoretic measures (like Lemma 1.4 does) or the significance of having short blocks (like Lemma 1.6 does).

1.4 Our results

We remedy the above state of affairs by providing a new, more general notion of KL-hardness that allows us to obtain next-block inaccessible entropy in a modular way while also encompassing what is needed for next-block pseudoentropy.

Like in KL-hardness for sampling, we will consider a pair of jointly distributed random variables (Y, X). Following the spirit of accessible entropy, the adversary $\widetilde{\mathsf{G}}$ for our new notion will try to *generate* Y together with X, rather than taking Y as input. That is, $\widetilde{\mathsf{G}}$ will take randomness $\widetilde{R}$ and output a pair $(\widetilde{Y}, \widetilde{X}) = \widetilde{\mathsf{G}}(\widetilde{R}) = (\widetilde{\mathsf{G}}_1(\widetilde{R}), \widetilde{\mathsf{G}}_2(\widetilde{R}))$, which we require to be always within the support of (Y, X). Note that $\widetilde{\mathsf{G}}$ need not be an online generator; it can generate both $\widetilde{Y}$ and $\widetilde{X}$ using the same randomness $\widetilde{R}$. Of course, if (Y, X) is efficiently samplable (as it would be in most cryptographic applications), $\widetilde{\mathsf{G}}$ could generate $(\widetilde{Y}, \widetilde{X})$ identically distributed to (Y, X) by just using the "honest" sampler G for (Y, X). So, in addition, we require that the adversary $\widetilde{\mathsf{G}}$ also come with a *simulator* S, that can simulate its coin tosses given only $\widetilde{Y}$. The goal of the adversary is to minimize the KL divergence

$$\mathrm{KL}\left(\widetilde{R}, \widetilde{Y} \,\middle\|\, \mathsf{S}(Y), Y\right)$$

for a uniformly random $\widetilde{R}$. This divergence measures both how well $\widetilde{\mathsf{G}}_1$ approximates the distribution of Y as well as how well S simulates the corresponding coin tosses of $\widetilde{\mathsf{G}}_1$. Note that when $\widetilde{\mathsf{G}}$ is the honest sampler G, the task of S is exactly to sample from the conditional distribution of $\widetilde{R}$ given $\mathsf{G}(\widetilde{R}) = Y$. However, the adversary may reduce the divergence by instead designing the sampler $\widetilde{\mathsf{G}}$ and simulator S to work in concert, potentially trading off how well $\widetilde{\mathsf{G}}(\widetilde{R})$ approximates Y in exchange for easier simulation by S. Explicitly, the definition is as follows.

Definition 1.9 (KL-hard, informal version of Definition 3.2)**.** *Let n be a security parameter, and (Y, X) be a pair of random variables jointly distributed over strings of length* $\mathrm{poly}(n)$*. We say that (Y, X) is* Δ-KL-hard *if the following holds.*

Let $\widetilde{\mathsf{G}} = (\widetilde{\mathsf{G}}_1, \widetilde{\mathsf{G}}_2)$ and S be probabilistic $\mathrm{poly}(n)$*-time algorithms such that* $\mathrm{Supp}(\widetilde{\mathsf{G}}(\widetilde{R})) \subseteq \mathrm{Supp}((Y, X))$*, where $\widetilde{R}$ is uniformly distributed. Then writing $\widetilde{Y} = \widetilde{\mathsf{G}}_1(\widetilde{R})$, we require that*

$$\mathrm{KL}\left(\widetilde{R}, \widetilde{Y} \,\middle\|\, \mathsf{S}(Y), Y\right) \geq \Delta.$$

Similarly to Lemma 1.4, we can show that one-way functions achieve this notion of KL hardness.

Lemma 1.10. *Let $f : \{0,1\}^n \to \{0,1\}^n$ be a one-way function and let X be uniformly distributed in $\{0,1\}^n$. Then $(f(X), X)$ is $\omega(\log n)$-KL-hard.*

Note that this lemma implies Lemma 1.4. If we take $\widetilde{\mathsf{G}}$ to be the "honest" sampler $\widetilde{\mathsf{G}}(x) = (f(x), x)$, then we have:

$$\mathrm{KL}\left(X, f(X) \,\|\, \mathsf{S}(Y), Y\right) = \mathrm{KL}\left(\widetilde{R}, \widetilde{Y} \,\middle\|\, \mathsf{S}(Y), Y\right),$$

which is is $\omega(\log n)$ by Lemma 1.10. That is, KL-hardness for sampling preimages (as in Definition 1.3 and Lemma 1.4) is obtained by fixing $\widetilde{\mathsf{G}}$ and focusing on the hardness for the simulator S, i.e. the divergence $\mathrm{KL}\left(\widetilde{Y} \,\middle\|\, Y\right)$.

Conversely, we show that inaccessible entropy comes by removing the simulator S from the definition, and focusing on the hardness for the generator $\widetilde{\mathsf{G}}$. It turns out that this removal is possible when we break Y into short blocks, yielding the following definition.

Definition 1.11 (next-block-KL-hard, informal)**.** *Let n be a security parameter, and $Y = (Y_1, \ldots, Y_m)$ be a random variable distributed on strings of length* $\mathrm{poly}(n)$*. We say that Y is* Δ-next-block-KL-hard *if the following holds.*

Let $\widetilde{\mathsf{G}}$ be any probabilistic $\mathrm{poly}(n)$*-time algorithm that takes a sequence of uniformly random strings $\widetilde{R} = (\widetilde{R}_1, \ldots, \widetilde{R}_m)$ and outputs a sequence $\widetilde{Y} = (\widetilde{Y}_1, \ldots, \widetilde{Y}_m)$ in an "online fashion" by which we mean that $\widetilde{Y}_i = \widetilde{\mathsf{G}}(\widetilde{R}_1, \ldots, \widetilde{R}_i)$ depends on only the first i random strings of $\widetilde{\mathsf{G}}$ for $i = 1, \ldots, m$. Suppose further that* $\mathrm{Supp}(\widetilde{Y}) \subseteq \mathrm{Supp}(Y)$*.*

We require that for all such $\widetilde{\mathsf{G}}$, we have:

$$\sum_{i=1}^{m} \mathrm{KL}\left(\widetilde{Y}_i | \widetilde{R}_{<i}, \widetilde{Y}_{<i} \,\middle\|\, Y_i | R_{<i}, Y_{<i}\right) \geq \Delta,$$

where we use the notation $z_{<i} = (z_1, \ldots, z_{i-1})$ and $R = (R_1, \ldots, R_m)$ is a dummy random variable independent of Y.

That is, the goal of the online generator $\widetilde{\mathsf{G}}$ is to generate $\widetilde{Y}_i$ given the history of coin tosses $\widetilde{R}_{<i}$ with the same conditional distribution as Y_i given $Y_{<i}$. Notice that there is no explicit simulator in the definition of next-block-KL-hardness. Nevertheless we can obtain it from KL-hardness by using sufficiently short blocks:

Lemma 1.12. *Let n be a security parameter, let (Y, X) be random variables distributed on strings of length* $\mathrm{poly}(n)$*, and let $Y = (Y_1, \ldots, Y_m)$ be a partition of Y into blocks of length $O(\log n)$.*

If (Y, X) is Δ-KL-hard, then $(Y_1, \ldots, Y_m, X)$ is $(\Delta - \mathrm{negl}(n))$-next-block-KL-hard.

An intuition for the proof is that since the blocks are of logarithmic length, given Y_i we can simulate the corresponding coin tosses of $\widetilde{R}_i$ of $\widetilde{\mathsf{G}}$ by rejection sampling and succeed with high probability in $\mathrm{poly}(n)$ tries.

A nice feature of the definition of next-block-KL-hardness compared to inaccessible entropy is that it is meaningful even for non-flat random variables, as KL divergence is always nonnegative. Moreover, for flat random variables, it equals the inaccessible entropy:

Lemma 1.13. *Suppose $Y = (Y_1, \ldots, Y_m)$ is a flat random variable. Then Y is Δ-next-block-KL-hard if and only if Y has accessible entropy at most $\mathrm{H}(Y) - \Delta$.*

Intuitively, this lemma comes from the identity that if Y is a flat random variable and $\mathrm{Supp}(\widetilde{Y}) \subseteq \mathrm{Supp}(Y)$, then $\mathrm{H}\left(\widetilde{Y}\right) = \mathrm{H}(Y) - \mathrm{KL}\left(\widetilde{Y} \,\middle\|\, Y\right)$. We stress that we do not require the individual blocks Y_i have flat distributions, only that the random variable Y as a whole is flat. For example, if f is a function and X is uniform, then $(f(X), X)$ is flat even though $f(X)$ itself may be far from flat.

Putting together Lemmas 1.10, 1.12, and 1.13, we obtain a new, more modular proof of Theorem 1.8. The reduction implicit in the combination of these lemmas is the same as the one in [HRVW09], but the analysis is different. (In particular, [HRVW09] makes no use of KL divergence.) Like the existing proof of Theorem 1.2, this proof separates the move from one-wayness to a form of KL-hardness, the role of short blocks, and the move from KL-hardness to computational entropy. Moreover, this further illumination of and toolkit for notions of computational entropy may open the door to other applications in cryptography.

We remark that another interesting direction for future work is to find a construction of universal one-way hash functions (UOWHFs) from one-way functions that follows a similar template to the above constructions of PRGs and SHCs. There is now a construction of UOWHFs based on a variant of inaccessible entropy [HHR+10], but it remains more complex and inefficient than those of PRGs and SHCs.

2 Preliminaries

Notations. For a tuple $x = (x_1, \ldots, x_n)$, we write $x_{\le i}$ for $(x_1, \ldots, x_i)$, and $x_{<i}$ for $(x_1, \ldots, x_{i-1})$.

poly denotes the set of polynomial functions and negl the set of all negligible functions: $\varepsilon \in \mathrm{negl}$ if for all $p \in \mathrm{poly}$ and large enough $n \in \mathbb{N}$, $\varepsilon(n) \le 1/p(n)$. We will sometimes abuse notations and write $\mathrm{poly}(n)$ to mean $p(n)$ for some $p \in \mathrm{poly}$ and similarly for $\mathrm{negl}(n)$.

PPT stands for probabilistic polynomial time and can be either in the uniform or non-uniform model of computation. All our results are stated as uniform polynomial time oracle reductions and are thus meaningful in both models.

For a random variable X over $\mathcal{X}$, $\mathrm{Supp}(X) \stackrel{\text{def}}{=} \{x \in \mathcal{X} : \Pr[X = x] > 0\}$ denotes the support of X. A random variable is *flat* if it is uniform over its support. Random variables will be written with uppercase letters and the associated lowercase letter represents a generic element from its support.

Information theory.

Definition 2.1 (Entropy)**.** *For a random variable X and $x \in \mathrm{Supp}(X)$, the* sample entropy *(also called surprise) of x is* $\mathrm{H}^*_x(X) \stackrel{\text{def}}{=} \log(1/\Pr[X = x])$*. The* entropy $\mathrm{H}(X)$ *of X is the expected sample entropy:* $\mathrm{H}(X) \stackrel{\text{def}}{=} \mathrm{E}_{x \leftarrow X}[\mathrm{H}^*_x(X)]$*.*

Definition 2.2 (Conditional entropy)**.** *Let (A, X) be a pair of random variables and consider $(a, x) \in \mathrm{Supp}(A, X)$, the* conditional sample entropy *of (a, x) is* $\mathrm{H}^*_{a,x}(A|X) \stackrel{\text{def}}{=} \log(1/\Pr[A = a \mid X = x])$ *and the* conditional entropy *of A given X is the expected conditional sample entropy:*

$$\mathrm{H}(A|X) \stackrel{\text{def}}{=} \mathop{\mathrm{E}}_{(a,x)\leftarrow(A,X)}\left[\log \frac{1}{\Pr[A = a \mid X = x]}\right] .$$

Proposition 2.3 (Chain rule for entropy)**.** *Let (A, X) be a pair of random variables, then* $\mathrm{H}(A, X) = \mathrm{H}(A|X) + \mathrm{H}(X)$ *and for $(a, x) \in \mathrm{Supp}(A, X)$,* $\mathrm{H}^*_{a,x}(A, X) = \mathrm{H}^*_{a,x}(A|X) + \mathrm{H}^*_x(X)$*.*

Definition 2.4 (KL-divergence)**.** *For a pair (A, B) of random variables and $(a, b) \in \mathrm{Supp}(A, B)$ the* sample KL-divergence *(*log*-probability ratio) is:*

$$\mathrm{KL}^*_a(A \,\|\, B) \stackrel{\text{def}}{=} \log \frac{\Pr[A = a]}{\Pr[B = a]} ,$$

and the KL-divergence *between A and B is the expected sample* KL*-divergence:*

$$\mathrm{KL}(A \,\|\, B) \stackrel{\text{def}}{=} \mathop{\mathrm{E}}_{a\leftarrow A}\left[\log \frac{\Pr[A = a]}{\Pr[B = a]}\right] .$$

Definition 2.5 (Conditional KL-divergence)**.** *For pairs of random variables (A, X) and (B, Y), and $(a, x) \in \mathrm{Supp}(A, X)$, the* conditional sample KL-divergence *is:*

$$\mathrm{KL}^*_{a,x}(A|X \,\|\, B|Y) \stackrel{\text{def}}{=} \log \frac{\Pr[A = a|X = x]}{\Pr[B = a|Y = x]} ,$$

and the conditional KL-divergence *is:*

$$\mathrm{KL}(A|X \,\|\, B|Y) \stackrel{\text{def}}{=} \mathop{\mathrm{E}}_{(a,x)\leftarrow(A,X)}\left[\log \frac{\Pr[A = a|X = x]}{\Pr[B = a|Y = x]}\right] .$$

Proposition 2.6 (Chain rule for KL-divergence)**.** *For pairs of random variables (X, A) and (Y, B):*

$$\mathrm{KL}(A, X \,\|\, B, Y) = \mathrm{KL}(A|X \,\|\, B|Y) + \mathrm{KL}(X \,\|\, Y) ,$$

and for $(a, x) \in \mathrm{Supp}(A, X)$:

$$\mathrm{KL}^*_{a,x}(A, X \,\|\, B, Y) = \mathrm{KL}^*_{a,x}(A|X \,\|\, B|Y) + \mathrm{KL}^*_x(X \,\|\, Y) .$$

Proposition 2.7 (Data-processing inequality)**.** *Let (X, Y) be a pair of random variables and let f be a function defined on* $\mathrm{Supp}(Y)$*, then:*

$$\mathrm{KL}\,(X \,\|\, Y) \geq \mathrm{KL}\,(f(X) \,\|\, f(Y)) \ .$$

Definition 2.8 (Max-KL-divergence)**.** *Let (X, Y) be a pair of random variables and $\delta \in [0, 1]$. We define $\mathrm{KL}^{\delta}_{\max}\,(X \,\|\, Y)$ to be the quantile of level δ of $\mathrm{KL}^*_x\,(X \,\|\, Y)$, equivalently it is the smallest $\Delta \in \mathbb{R}$ satisfying:*

$$\Pr_{x \leftarrow X}\left[\mathrm{KL}^*_x\,(X \,\|\, Y) \leq \Delta\right] \geq \delta \ ,$$

and it is characterized by the following equivalence:

$$\mathrm{KL}^{\delta}_{\max}\,(X \,\|\, Y) > \Delta \iff \Pr_{x \leftarrow X}\left[\mathrm{KL}^*_x\,(X \,\|\, Y) \leq \Delta\right] < \delta \ .$$

Block generators

Definition 2.9 (Block generator)**.** *An* m-block generator *is a function $\mathsf{G} : \{0,1\}^s \to \prod_{i=1}^{m}\{0,1\}^{\ell_i}$. $\mathsf{G}_i(r)$ denotes the i-th block of G on input r and $|\mathsf{G}_i| = \ell_i$ denotes the bit length of the i-th block.*

Definition 2.10 (Online generator)**.** *An* online m-block generator *is a function $\widetilde{\mathsf{G}} : \prod_{i=1}^{m}\{0,1\}^{s_i} \to \prod_{i=1}^{m}\{0,1\}^{\ell_i}$ such that for all $i \in [m]$ and $r \in \prod_{i=1}^{m}\{0,1\}^{s_i}$, $\widetilde{\mathsf{G}}_i(r)$ only depends on $r_{\leq i}$. We sometimes write $\widetilde{\mathsf{G}}_i(r_{\leq i})$ when the input blocks $i+1, \ldots, m$ are unspecified.*

Definition 2.11 (Support)**.** *The* support *of a generator G is the support of the random variable $\mathrm{Supp}\big(\mathsf{G}(R)\big)$ for uniform input R. If G is an $(m+1)$-block generator, and Π is a binary relation, we say that G is* supported on Π *if $\mathrm{Supp}\big(\mathsf{G}_{\leq m}(R), \mathsf{G}_{m+1}(R)\big) \subseteq \Pi$.*

When G is an $(m+1)$-block generator supported on a binary relation Π, we will often use the notation $\mathsf{G}_{\mathsf{w}} \stackrel{\text{def}}{=} \mathsf{G}_{m+1}$ to emphasize that the last block corresponds to a witness for the first m blocks.

Cryptography.

Definition 2.12 (One-way Function)**.** *Let n be a security parameter, $t = t(n)$ and $\varepsilon = \varepsilon(n)$. A function $f : \{0,1\}^n \to \{0,1\}^n$ is a (t, ε)-*one-way function *if:*

1. *For all time t randomized algorithm A: $\Pr_{x \leftarrow U_n}\left[\mathsf{A}\big(f(x)\big) \in f^{-1}\big(f(x)\big)\right] \leq \varepsilon$, where U_n is uniform over $\{0,1\}^n$.*
2. *There exists a* PPT *algorithm B such that $\mathsf{B}(x, 1^n) = f(x)$ for all $x \in \{0,1\}^n$.*

If f is $(n^c, 1/n^c)$-one-way for every $c \in \mathbb{N}$, we say that f is (strongly) one-way*.*

3 Search Problems and KL-hardness

In this section, we first present the classical notion of hard-on-average search problems and introduce the new notion of KL-hardness. We then relate the two notions by proving that average-case hardness implies KL-hardness.

3.1 Search problems

For a binary relation $\Pi \subseteq \{0,1\}^* \times \{0,1\}^*$, we write $\Pi(y,w)$ for the predicate that is true iff $(y,w) \in \Pi$ and say that w is a *witness* for the *instance* y[2]. To each relation Π, we naturally associate (1) a *search problem*: given y, find w such that $\Pi(y,w)$ or state that no such w exist and (2) the *decision problem* defined by the language $L_\Pi \stackrel{\text{def}}{=} \{y \in \{0,1\}^* : \exists w \in \{0,1\}^*, \Pi(y,w)\}$. **FNP** denotes the set of all relations Π computable by a polynomial time algorithm and such that there exists a polynomial p such that $\Pi(y,w) \Rightarrow |w| \le p(|y|)$. Whenever $\Pi \in \mathbf{FNP}$, the associated decision problem L_Π is in **NP**. We now define average-case hardness.

Definition 3.1 (Distributional search problem)**.** *A* distributional search problem *is a pair* (Π, Y) *where* $\Pi \subseteq \{0,1\}^* \times \{0,1\}^*$ *is a binary relation and* Y *is a random variable supported on* L_Π*.*

The problem (Π, Y) *is* (t, ε)-hard *if* $\Pr\big[\Pi\big(Y, \mathsf{A}(Y)\big)\big] \le \varepsilon$ *for all time* t *randomized algorithm* A*, where the probability is over the distribution of* Y *and the randomness of* A*.*

Example. For $f : \{0,1\}^n \to \{0,1\}^n$, the problem of inverting f is the search problem associated with the relation $\Pi^f \stackrel{\text{def}}{=} \{(f(x), x) : x \in \{0,1\}^n\}$. If f is a (t,ε)-one-way function, then the distributional search problem $\big(\Pi^f, f(X)\big)$ of inverting f on a uniform random input $X \in \{0,1\}^n$ is (t,ε)-hard.

Remark. Consider a distributional search problem (Π, Y). Without loss of generality, there exists a (possibly inefficient) two-block generator $\mathsf{G} = (\mathsf{G}_1, \mathsf{G}_\mathsf{w})$ supported on Π such that $\mathsf{G}_1(R) = Y$ for uniform input R. If G_w is polynomial-time computable, it is easy to see that the search problem $\big(\Pi^{\mathsf{G}_1}, \mathsf{G}_1(R)\big)$ is at least as hard as (Π, Y). The advantage of writing the problem in this "functional" form is that the distribution $(\mathsf{G}_1(R), R)$ over (instance, witness) pairs is flat, which is a necessary condition to relate hardness to inaccessible entropy (see Theorem 4.6).

Furthermore, if G_1 is also polynomial-time computable and (Π, Y) is $(\mathrm{poly}(n), \mathrm{negl}(n))$-hard, then $R \mapsto \mathsf{G}_1(R)$ is a one-way function. Combined with the previous example, we see that the existence of one-way functions is equivalent to the existence of $(\mathrm{poly}(n), \mathrm{negl}(n))$-hard search problems for which (instance, witness) pairs can be efficiently sampled.

3.2 KL-hardness

Instead of considering an adversary directly attempting to solve a search problem (Π, Y), the adversary in the definition of KL-hardness comprises a pair of algorithm $(\widetilde{\mathsf{G}}, \mathsf{S})$ where $\widetilde{\mathsf{G}}$ is a two-block generator outputting valid (instance, witness) pairs for Π and S is a *simulator* for $\widetilde{\mathsf{G}}$: given an instance y, the goal of S is to output randomness r for $\widetilde{\mathsf{G}}$ such that $\widetilde{\mathsf{G}}_1(r) = y$. Formally, the definition is as follows.

Definition 3.2 (KL-hard)**.** *Let* (Π, Y) *be a distributional search problem. We say that* (Π, Y) *is* (t, Δ)-KL-hard *if:*

$$\mathrm{KL}\left(\widetilde{R}, \widetilde{\mathsf{G}}_1(\widetilde{R}) \,\middle\|\, \mathsf{S}(Y), Y\right) > \Delta ,$$

[2]We used the unconventional notation y for the instance (instead of x) because our relations will often be of the form Π^f for some function f; in this case an instance is some y in the range of f and a witness for y is any preimage $x \in f^{-1}(y)$.

for all pairs $(\widetilde{\mathsf{G}}, \mathsf{S})$ *of time* t *algorithms where* $\widetilde{\mathsf{G}}$ *is a two-block generator supported on* Π *and* $\widetilde{R}$ *is uniform randomness for* $\widetilde{\mathsf{G}}_1$. *Similarly,* (Π, Y) *is* (t, Δ)-$\mathrm{KL}^{\delta}_{\max}$-hard *if for all such pairs:*

$$\mathrm{KL}^{\delta}_{\max}\left(\widetilde{R}, \widetilde{\mathsf{G}}_1(\widetilde{R}) \,\middle\|\, \mathsf{S}(Y), Y\right) > \Delta\ .$$

Note that a pair $(\widetilde{\mathsf{G}}, \mathsf{S})$ achieves a KL-divergence of zero in Definition 3.2 if $\widetilde{\mathsf{G}}_1(R)$ has the same distribution as Y and if $\widetilde{\mathsf{G}}_1\big(\mathsf{S}(y)\big) = y$ for all $y \in \mathrm{Supp}(Y)$. In this case, writing $\widetilde{\mathsf{G}}_{\mathsf{w}} \stackrel{\text{def}}{=} \widetilde{\mathsf{G}}_2$, we have that $\widetilde{\mathsf{G}}_{\mathsf{w}}\big(\mathsf{S}(Y)\big)$ is a valid witness for Y since $\widetilde{\mathsf{G}}$ is supported on Π.

More generally, the composition $\widetilde{\mathsf{G}}_{\mathsf{w}} \circ \mathsf{S}$ solves the search problem (Π, Y) whenever $\widetilde{\mathsf{G}}_1\big(\mathsf{S}(Y)\big) = Y$. When the KL-divergences in Definition 3.2 are upper-bounded, we can lower bound the probability of the search problem being solved (Lemma 3.4) This immediately implies that hard search problems are also KL-hard.

Theorem 3.3. *Let* (Π, Y) *be a distributional search problem. If* (Π, Y) *is* (t, ε)*-hard, then it is* (t', Δ')-KL*-hard and* (t', Δ'')-$\mathrm{KL}^{\delta}_{\max}$*-hard for every* $\delta \in [0, 1]$ *where* $t' = \Omega(t)$,[3] $\Delta' = \log(1/\varepsilon)$ *and* $\Delta'' = \log(1/\varepsilon) - \log(1/\delta)$.

Remark. As we see, a "good" simulator S for a generator $\widetilde{\mathsf{G}}$ is one for which $\widetilde{\mathsf{G}}_1\big(\mathsf{S}(Y)\big) = Y$ holds often. It will be useful in Section 4 to consider simulators S which are allowed to fail by outputting a failure string $r \notin \mathrm{Supp}(\widetilde{R})$, (e.g. $r = \bot$) and adopt the convention that $\widetilde{\mathsf{G}}_1(r) = \bot$ whenever $r \notin \mathrm{Supp}(\widetilde{R})$. With this convention, we can without loss of generality add the requirement that $\widetilde{\mathsf{G}}_1\big(\mathsf{S}(Y)\big) = Y$ whenever $\mathsf{S}(Y) \in \mathrm{Supp}(\widetilde{R})$: indeed, S can always check that it is the case and if not output a failure symbol. For such a simulator S, observe that for all $r \in \mathrm{Supp}(\widetilde{R})$, the second variable on both sides of the KL-divergences in Definition 3.2 is obtained by applying $\widetilde{\mathsf{G}}_1$ on the first variable and can thus be dropped, leading to a simpler definition of KL-hardness: $\mathrm{KL}\left(\widetilde{R} \,\middle\|\, \mathsf{S}(Y)\right) > \Delta$.

Theorem 3.3 is an immediate consequence of the following lemma.

Lemma 3.4. *Let* (Π, Y) *be a distributional search problem and* $(\widetilde{\mathsf{G}}, \mathsf{S})$ *be a pair of algorithms with* $\widetilde{\mathsf{G}} = (\widetilde{\mathsf{G}}_1, \widetilde{\mathsf{G}}_{\mathsf{w}})$ *a two-block generator supported on* Π. *Define the linear-time oracle algorithm* $\mathsf{A}^{\widetilde{\mathsf{G}}_{\mathsf{w}}, \mathsf{S}}(y) \stackrel{\text{def}}{=} \widetilde{\mathsf{G}}_{\mathsf{w}}(\mathsf{S}(y))$. *For* $\Delta \in \mathbb{R}^+$ *and* $\delta \in [0, 1]$*:*

1. *If* $\mathrm{KL}\left(\widetilde{R}, \widetilde{\mathsf{G}}_1(\widetilde{R}) \,\middle\|\, \mathsf{S}(Y), Y\right) \le \Delta$ *then* $\Pr\left[\Pi(Y, \mathsf{A}^{\widetilde{\mathsf{G}}_{\mathsf{w}}, \mathsf{S}}(Y))\right] \ge 1/2^{\Delta}$.
2. *If* $\mathrm{KL}^{\delta}_{\max}\left(\widetilde{R}, \widetilde{\mathsf{G}}_1(\widetilde{R}) \,\middle\|\, \mathsf{S}(Y), Y\right) \le \Delta$ *then* $\Pr\left[\Pi(Y, \mathsf{A}^{\widetilde{\mathsf{G}}_{\mathsf{w}}, \mathsf{S}}(Y))\right] \ge \delta/2^{\Delta}$.

[3] For the theorems in this paper that relate two notions of hardness, the notation $t' = \Omega(t)$ means that there exists a constant C depending *only* on the computational model such that $t' \ge C \cdot t$.

Proof. We have:

$$
\begin{aligned}
\Pr\left[\Pi\big(Y, \mathsf{A}^{\widetilde{\mathsf{G}}_{\mathsf{w}},\mathsf{S}}(Y)\big)\right] &= \Pr\left[\Pi(Y, \widetilde{\mathsf{G}}_{\mathsf{w}}(\mathsf{S}(Y)))\right] \\
&\geq \Pr\left[\widetilde{\mathsf{G}}_1(\mathsf{S}(Y)) = Y\right] \qquad\qquad (\widetilde{\mathsf{G}} \text{ is supported on } \Pi) \\
&= \sum_{r \in \mathrm{Supp}(\widetilde{R})} \Pr\left[\mathsf{S}(Y) = r \wedge Y = \widetilde{\mathsf{G}}_1(r)\right] \\
&= \mathop{\mathrm{E}}_{r \leftarrow \widetilde{R}}\left[\frac{\Pr\left[\mathsf{S}(Y) = r \wedge Y = \widetilde{\mathsf{G}}_1(r)\right]}{\Pr\left[\widetilde{R} = r\right]}\right] \\
&= \mathop{\mathrm{E}}_{\substack{r \leftarrow \widetilde{R} \\ y \leftarrow \widetilde{\mathsf{G}}_1(r)}}\left[2^{-\mathrm{KL}^*_{r,y}\left(\widetilde{R}, \widetilde{\mathsf{G}}_1(\widetilde{R}) \,\|\, \mathsf{S}(Y), Y\right)}\right] .
\end{aligned}
$$

Now, the first claim follows by Jensen's inequality (since $x \mapsto 2^{-x}$ is convex) and the second claim follows by Markov' inequality when considering the event that the sample-KL is smaller than Δ (which occurs with probability at least δ by assumption). $\square$

Relation to KL-hardness for sampling. In [VZ12], the authors introduced the notion of KL-hardness for sampling: for jointly distributed variables (Y, W), W is hard for sampling given Y if it is hard for a polynomial time adversary to approximate—measured in KL-divergence—the conditional distribution W given Y. Formally:

Definition 3.5 (KL-hard for sampling, Def. 3.4 in [VZ12])**.** *Let (Y, W) be a pair of random variables, we say that W is (t, Δ)-*KL-hard to sample given *Y if for all time t randomized algorithm S, we have:*

$$\mathrm{KL}\left(Y, W \,\|\, Y, \mathsf{S}(Y)\right) > \Delta\,.$$

As discussed in Section 1.2, it was shown in [VZ12] that if $f : \{0,1\}^n \to \{0,1\}^n$ is a one-way function, then $\big(f(X), X_1, \ldots, X_n\big)$ has next-bit pseudoentropy for uniform $X \in \{0,1\}^n$ (see Theorem 1.2). The first step in proving this result was to prove that X is KL-hard to sample given $f(X)$ (see Lemma 1.4).

We observe that when (Y, W) is of the form $(f(X), X)$ for some function $f : \{0,1\}^n \to \{0,1\}^n$ and variable X over $\{0,1\}^n$, then KL-hardness for sampling is implied by KL-hardness by simply fixing $\widetilde{\mathsf{G}}$ to be the "honest sampler" $\widetilde{\mathsf{G}}(X) = (f(X), X)$. Indeed, in this case we have:

$$\mathrm{KL}\left(X, \widetilde{\mathsf{G}}_1(X) \,\middle\|\, \mathsf{S}(Y), Y\right) = \mathrm{KL}\left(X, f(X) \,\|\, \mathsf{S}(Y), Y\right)\,.$$

We can thus recover Lemma 1.4 as a direct corollary of Theorem 3.3.

Corollary 3.6. *Consider a function $f : \{0,1\}^n \to \{0,1\}^n$ and define $\Pi^f \stackrel{\text{def}}{=} \{(f(x), x) : x \in \{0,1\}^n\}$ and $Y \stackrel{\text{def}}{=} f(X)$ for X uniform over $\{0,1\}^n$. If f is (t, ε)-one-way, then (Π^f, Y) is $\big(t', \log(1/\varepsilon)\big)$-*KL*-hard and X is $\big(t', \log(1/\varepsilon)\big)$-*KL*-hard to sample given Y with $t' = \Omega(t)$.*

Witness KL-hardness. We also introduce a relaxed notion of KL-hardness called witness-KL-hardness. In this notion, we further require $(\widetilde{\mathsf{G}}, \mathsf{S})$ to approximate the joint distribution of (instance, witness) pairs rather than only instances. For example, the problem of inverting a function$f$ over a random input X is naturally associated with the distribution $\big(f(X), X\big)$. The relaxation in this case is analogous to the notion of *distributional one-way function* for which the adversary is required to approximate the uniform distribution over preimages.

Definition 3.7 (Witness KL-hardness)**.** *Let* Π *be a binary relation and* (Y, W) *be a pair of random variables supported on* Π*. We say that* (Π, Y, W) *is* (t, Δ)-witness-KL-hard *if for all pairs of time* t *algorithms* $(\widetilde{\mathsf{G}}, \mathsf{S})$ *where* $\widetilde{\mathsf{G}}$ *is a two-block generator supported on* Π*, for uniform* $\widetilde{R}$*:*

$$\mathrm{KL}\left(\widetilde{R}, \widetilde{\mathsf{G}}_1(\widetilde{R}), \widetilde{\mathsf{G}}_{\mathsf{w}}(\widetilde{R}) \,\middle\|\, \mathsf{S}(Y), Y, W\right) > \Delta \,.$$

Similarly, for $\delta \in [0,1]$*,* (Π, Y, W) *is* (t, Δ)-witness-$\mathrm{KL}^{\delta}_{\max}$-hard*, if for all such pairs:*

$$\mathrm{KL}^{\delta}_{\max}\left(\widetilde{R}, \widetilde{\mathsf{G}}_1(\widetilde{R}), \widetilde{\mathsf{G}}_{\mathsf{w}}(\widetilde{R}) \,\middle\|\, \mathsf{S}(Y), Y, W\right) > \Delta \,.$$

We introduced KL-hardness first, since it is the notion which is most directly obtained from the hardness of distribution search problems. Observe that by the data processing inequality for KL divergence (Proposition 2.7), dropping the third variable on both sides of the KL divergences in Definition 3.7 only decreases the divergences. Hence, KL-hardness implies witness-KL-hardness as stated in (Theorem 3.8). As we will see in Section 4 witness-KL-hardness is the “correct” notion to obtain inaccessible entropy from: it is in fact equal to inaccessible entropy up to $1/\operatorname{poly}$ losses.

Theorem 3.8. *Let* Π *be a binary relation and* (Y, W) *be a pair of random variables supported on* Π*. If* (Π, Y) *is* (t, ε)*-hard, then* (Π, Y, W) *is* (t', Δ')*-witness-*KL*-hard and* (t', Δ'')*-witness-*$\mathrm{KL}^{\delta}_{\max}$*-hard for every* $\delta \in [0,1]$ *where* $t' = \Omega(t)$*,* $\Delta' = \log(1/\varepsilon)$ *and* $\Delta'' = \log(1/\varepsilon) - \log(1/\delta)$*.*

Remark. The data processing inequality does not hold exactly for $\mathrm{KL}_{\max}$, hence the $\mathrm{KL}_{\max}$ statement in Theorem 3.8 does not follow with the claimed parameters in a black-box manner from Theorem 3.3. However, an essentially identical proof given in Appendix A yields the result.

4 Inaccessible Entropy and Witness KL-hardness

In this section, we relate our notion of witness KL-hardness to the inaccessible entropy definition of [HRVW16]. Roughly speaking, we “split” the KL-hard definition into blocks to obtain an intermediate blockwise KL-hardness property (Section 4.1) that we then relate to inaccessible entropy (Section 4.2). Together, these results show that if f is a one-way function, the generator $\mathsf{G}^f(X) = \big(f(X)_1, \ldots, f(X)_n, X\big)$ has superlogarithmic inaccessible entropy.

4.1 Next-block KL-hardness and rejection sampling

Consider a binary relation Π and a pair of random variables (Y, W) supported on Π. For an online $(m+1)$-block generator $\widetilde{\mathsf{G}}$ supported on Π, it is natural to consider the simulator $\mathsf{Sim}_T^{\widetilde{\mathsf{G}}}$ that exploits the block structure of $\widetilde{\mathsf{G}}$: on input $Y \stackrel{\text{def}}{=} (Y_1, \ldots, Y_m)$, $\mathsf{Sim}_T^{\widetilde{\mathsf{G}}}(Y)$ generates randomness $\widehat{R} = (\widehat{R}_1, \ldots, \widehat{R}_m)$ block by block using rejection sampling until $\widetilde{\mathsf{G}}_i(\widehat{R}_{\leq i}) = Y_i$. The subscript T is the maximum number of attempts after which $\mathsf{Sim}_T^{\widetilde{\mathsf{G}}}$ gives up and outputs $\perp$. The formal definition of $\mathsf{Sim}_T^{\widetilde{\mathsf{G}}}$ is given in Algorithm 1.

Algorithm 1 Rejection sampling simulator $\mathsf{Sim}_T^{\widetilde{\mathsf{G}}}$

```
Input: y_1, ..., y_m ∈ ({0,1}^*)^m
Output: r̂_1, ..., r̂_m ∈ ({0,1}^v ∪ {⊥})^m
  for i = 1 to m do
      repeat
          sample r̂_i ← {0,1}^v
      until G̃_i(r̂_{≤i}) = y_i or ≥ T attempts
      if G̃_i(r̂_{≤i}) ≠ y_i then
          r̂_j ← ⊥ for j ≥ i; return
      end if
  end for
```

As will be established in Theorem 4.3, the "approximation error" of the pair $(\widetilde{\mathsf{G}}, \mathsf{Sim}_T^{\widetilde{\mathsf{G}}})$, namely $\mathrm{KL}\left(\widehat{R}, \widetilde{\mathsf{G}}_{\leq m}(\widehat{R}), \widetilde{\mathsf{G}}_{\mathsf{w}}(\widehat{R}) \,\middle\|\, \mathsf{Sim}_T^{\widetilde{\mathsf{G}}}(Y), Y, W\right)$, decomposes as the sum of two terms:

1. The first term measures how well $\widetilde{\mathsf{G}}_{\leq m}$ approximates the distribution Y in an online manner.

2. The second term measures the success probability of the rejection sampling procedure.

The second term can be made arbitrarily small by setting the number of trials T in $\mathsf{Sim}_T^{\widetilde{\mathsf{G}}}$ to be a large enough multiple of $m \cdot 2^{\ell}$ where ℓ is the length of the blocks of $\widetilde{\mathsf{G}}_{\leq m}$ (Lemma 4.4). This leads to a $\mathrm{poly}(m)$ time algorithm whenever ℓ is logarithmic in m. That is, given an online block generator $\widetilde{\mathsf{G}}$ for which $\widetilde{\mathsf{G}}_{\leq m}$ has short blocks, we obtain a corresponding simulator "for free". This leads to the following clean definition of next-block hardness that makes no reference to simulators.

Definition 4.1 (Next-block KL-hardness)**.** *The joint distribution* $(Y_1, \ldots, Y_{m+1})$ *is* (t, Δ)-block-KL-hard*, if for every time t online $(m+1)$-block generator $\widetilde{\mathsf{G}}$ supported on $Y_{\leq m+1}$, writing $\widetilde{Y}_{\leq m+1} \stackrel{\text{def}}{=} \widetilde{\mathsf{G}}(\widetilde{R}_{\leq m+1})$ for uniform $\widetilde{R}_{\leq m+1}$, we have:*

$$\sum_{i=1}^{m+1} \mathrm{KL}\left(\widetilde{Y}_i | \widetilde{R}_{<i}, \widetilde{Y}_{<i} \,\middle\|\, Y_i | R_{<i}, Y_{<i}\right) > \Delta\,,$$

where R_i is a "dummy" random variable over the domain of $\widetilde{\mathsf{G}}_i$ and independent of $Y_{\leq m+1}$. Similarly, for $\delta \in [0,1]$, we say that $(Y_1, \ldots, Y_{m+1})$ is (t, Δ)-block-$\mathrm{KL}^{\delta}_{\max}$-hard *if for every*

$\widetilde{\mathsf{G}}$ *as above:*

$$\Pr_{\substack{r_{\le m+1} \leftarrow \widetilde{R}_{\le m+1} \\ y_{\le m+1} \leftarrow \widetilde{\mathsf{G}}(r_{\le m+1})}} \left[\sum_{i=1}^{m+1} \mathrm{KL}^*_{y_i, r_{<i}, y_{<i}} \left(\widetilde{Y}_i | \widetilde{R}_{<i}, \widetilde{Y}_{<i} \, \middle\| \, Y_i | R_{<i}, Y_{<i} \right) \le \Delta \right] < \delta \,,$$

where $(\widetilde{Y}_{\le m+1}, \widetilde{R}_{\le m+1})$ *are defined as above.*

Remark. Since $\widetilde{Y}_{<i}$ is a function of $\widetilde{R}_{<i}$, the first conditional distribution in the KL is effectively $\widetilde{Y}_i | \widetilde{R}_{<i}$. Similarly the second distribution is effectively $Y_i | Y_{<i}$. The extra random variables are there for syntactic consistency.

We are now ready to present the main result of this section — witness-KL-hardness implies next-block KL-hardness — which is established by decomposing:

$$\mathrm{KL} \left(\widetilde{R}, \widetilde{\mathsf{G}}_{\le m}(\widetilde{R}), \widetilde{\mathsf{G}}_{\mathsf{w}}(\widetilde{R}) \, \middle\| \, \mathsf{Sim}^{\widetilde{\mathsf{G}}}_T(Y), Y, W \right)$$

as outlined above.

Theorem 4.2. *Let* Π *be a binary relation and let* (Y, W) *be a pair of random variables supported on* Π*. Let* $Y = (Y_1, \ldots, Y_m)$ *where the bit length of* Y_i *is at most* ℓ*. Then we have:*

1. *if* (Π, Y, W) *is* (t, Δ)*-witness-*KL*-hard, then for every* $0 < \Delta' \le \Delta$*,* $(Y_1, \ldots, Y_m, W)$ *is* $(t', \Delta - \Delta')$*-block-*KL*-hard where* $t' = \Omega(t\Delta'/(m^2 2^\ell))$*.*

2. *if* (Π, Y, W) *is* (t, Δ)*-witness-*$\mathrm{KL}^\delta_{\max}$*-hard, then for every* $0 < \Delta' \le \Delta$ *and* $0 \le \delta' \le 1 - \delta$*, we have that* $(Y_1, \ldots, Y_m, W)$ *is* $(t', \Delta - \Delta')$*-block-*$\mathrm{KL}^{\delta+\delta'}_{\max}$*-hard where* $t' = \Omega(t\delta'\Delta'/(m^2 2^\ell))$*.*

Proof. We apply the following lemma with $(Y_{\le m}, Y_{m+1}) \stackrel{\text{def}}{=} (Y, W)$. The theorem follows since when the oracle $\widetilde{\mathsf{G}}$ is replaced by an algorithm with running time t', without loss of generality the maximum length of an input random block is $v \le t'$ and the running time of the simulator becomes $O(mTt')$ for $T = m \cdot 2^\ell / (\Delta' \ln 2)$ in the witness-KL-hard case and $T = m \cdot 2^\ell / (\delta'\Delta' \ln 2)$ in the witness-$\mathrm{KL}_{\max}$-hard case. □

Lemma 4.3. *For a joint distribution* $(Y_1, \ldots, Y_{m+1})$*, let* $\widetilde{\mathsf{G}}$ *be an online* $(m+1)$*-block generator supported on* $Y_{\le m+1}$*. Define* $(\widetilde{Y}_1, \ldots, \widetilde{Y}_{m+1}) \stackrel{\text{def}}{=} \widetilde{\mathsf{G}}(\widetilde{R})$ *for uniform random variable* $\widetilde{R} = (\widetilde{R}_1, \ldots, \widetilde{R}_{m+1})$ *and let* R_i *be a "dummy" random variable over the domain of* $\widetilde{\mathsf{G}}_i$ *and independent of* $Y_{\le m+1}$*. If*

$$\sum_{i=1}^{m+1} \mathrm{KL} \left(\widetilde{Y}_i | \widetilde{R}_{<i}, \widetilde{Y}_{<i} \, \middle\| \, Y_i | R_{<i}, Y_{<i} \right) \le \Delta$$

then:

$$\mathrm{KL} \left(\widetilde{R}, \widetilde{\mathsf{G}}_{\le m}(\widetilde{R}), \widetilde{\mathsf{G}}_{m+1}(\widetilde{R}) \, \middle\| \, \mathsf{Sim}^{\widetilde{\mathsf{G}}}_T(Y), Y_{\le m}, Y_{m+1} \right) \le \Delta + \frac{m \cdot 2^\ell}{T \ln 2} \,.$$

Similarly, if

$$\Pr_{(y_{\le m+1}, r) \leftarrow (\widetilde{Y}_{\le m+1}, \widetilde{R})} \left[\sum_{i=1}^{m+1} \mathrm{KL}^*_{y,r} \left(\widetilde{Y}_i | \widetilde{R}_{<i}, \widetilde{Y}_{<i} \, \middle\| \, Y_i | R_{<i}, Y_{<i} \right) \le \Delta \right] \ge \delta,$$

then for all $\delta' \in (0, \delta]$:

$$\mathrm{KL}_{\max}^{\delta-\delta'}\left(\widetilde{R}, \widetilde{\mathsf{G}}(\widetilde{R}) \,\middle\|\, \mathsf{Sim}_T^{\widetilde{\mathsf{G}}}(Y), Y\right) \leq \Delta + \frac{m \cdot 2^{\ell}}{T\delta' \ln 2} .$$

Moreover, the running time of $\mathsf{Sim}_T^{\widetilde{\mathsf{G}}}$ is $O(mvT)$ with at most mT oracle calls to $\widetilde{\mathsf{G}}$, where v is the longest length of a random input r_i.

Proof. Define $\widehat{R} \stackrel{\text{def}}{=} \mathsf{Sim}_T^{\widetilde{\mathsf{G}}}(Y)$, and $\widehat{Y} \stackrel{\text{def}}{=} \widetilde{\mathsf{G}}(\widehat{R})$. We focus on sample notions first and consider $r \in \mathrm{Supp}(\widetilde{R})$ and $y \stackrel{\text{def}}{=} \widetilde{\mathsf{G}}(r)$. Then by the Chain Rule (Proposition 2.6),

$$\begin{aligned}
&\mathrm{KL}_{r,y}^{*}\left(\widetilde{R}, \widetilde{\mathsf{G}}(\widetilde{R}) \,\middle\|\, \mathsf{Sim}_T^{\widetilde{\mathsf{G}}}(Y), Y\right) = \mathrm{KL}_{r,y}^{*}\left(\widetilde{R}, \widetilde{Y} \,\middle\|\, \widehat{R}, \widehat{Y}\right) \\
&= \sum_{i=1}^{m}\left(\mathrm{KL}_{r,y}^{*}\left(\widetilde{R}_i|\widetilde{R}_{<i}, \widetilde{Y}_{\leq i} \,\middle\|\, \widehat{R}_i|\widehat{R}_{<i}, \widehat{Y}_{\leq i}\right) + \mathrm{KL}_{r,y}^{*}\left(\widetilde{Y}_i|\widetilde{R}_{<i}, \widetilde{Y}_{<i} \,\middle\|\, \widehat{Y}_i|\widehat{R}_{<i}, \widehat{Y}_{<i}\right)\right) \\
&= \sum_{i=1}^{m}\mathrm{KL}_{r,y}^{*}\left(\widetilde{Y}_i|\widetilde{R}_{<i}, \widetilde{Y}_{<i} \,\middle\|\, \widehat{Y}_i|\widehat{R}_{<i}, \widehat{Y}_{<i}\right) \\
&= \sum_{i=1}^{m}\mathrm{KL}_{r,y}^{*}\left(\widetilde{Y}_i|\widetilde{R}_{<i} \,\middle\|\, \widehat{Y}_i|\widehat{R}_{<i}\right) ,
\end{aligned}$$

The penultimate equality is by definition of rejection sampling: $\widetilde{R}_i|\widetilde{R}_{<i}, \widetilde{Y}_{\leq i}$ and $\widehat{R}_i|\widehat{R}_{<i}, \widehat{Y}_{\leq i}$ are identical on $\mathrm{Supp}(\widetilde{R}_i)$ since conditioning on $\widehat{Y}_i = y$ implies that only non-failure cases ($\widehat{R}_i \neq \perp$) are considered. The last equality is because $\widetilde{Y}_{<i}$ (resp. $\widehat{Y}_{<i}$) is a deterministic function of $\widetilde{R}_{<i}$ (resp. $\widehat{R}_{<i}$).

We now relate $\widehat{Y}_i|\widehat{R}_{<i}$ to $Y_i|Y_{<i}$:

$$\begin{aligned}
&\Pr\left[\widehat{Y}_i = y_i|\widehat{R}_{<i} = r_{<i}\right] \\
&\quad= \Pr\left[\widehat{Y}_i = y_i, Y_i = y_i|\widehat{R}_{<i} = r_{<i}\right] && (\widehat{Y}_i = y_i \Leftrightarrow \widehat{Y}_i = y_i \wedge Y_i = y_i) \\
&\quad= \Pr\left[\widehat{Y}_i = y_i|Y_i = y_i, \widehat{R}_{<i} = r_{<i}\right] \cdot \Pr\left[Y_i = y_i|\widehat{R}_{<i} = r_{<i}\right] && \text{(Bayes' Rule)} \\
&\quad= \Pr\left[\widehat{Y}_i = y_i|Y_i = y_i, \widehat{R}_{<i} = r_{<i}\right] \cdot \Pr\left[Y_i = y_i|Y_{<i} = y_{<i}\right] ,
\end{aligned}$$

where the last equality is because when $r \in \mathrm{Supp}(\widetilde{R})$, $\widehat{R}_{<i} = r_{<i} \Rightarrow Y_{<i} = y_{<i}$ and because Y_i is independent of $\widehat{R}_{<i}$ given $Y_{<i}$ (as $\widehat{R}_{<i}$ is simply a randomized function of $Y_{<i}$). Combining the previous two derivations we obtain:

$$\begin{aligned}
\mathrm{KL}_{r,y}^{*}\left(\widetilde{R}, \widetilde{\mathsf{G}}(\widetilde{R}) \,\middle\|\, \mathsf{Sim}_T^{\widetilde{\mathsf{G}}}(Y), Y\right) &= \sum_{i=1}^{m}\mathrm{KL}_{r,y}^{*}\left(\widetilde{Y}_i|\widetilde{R}_{<i}, \widetilde{Y}_{<i} \,\middle\|\, Y_i|R_{<i}, Y_{<i}\right) \\
&\quad+ \sum_{i=1}^{m}\log\left(\frac{1}{\Pr\left[\widehat{Y}_i = y_i|Y_i = y_i, \widehat{R}_{<i} = r_{<i}\right]}\right) .
\end{aligned}$$

Now, the first claim of the lemma follows by taking expectations on both sides and directly applying the following lemma:

Lemma 4.4. *Let $\widetilde{\mathsf{G}}$ be an online m-block generator, and let $L_i \stackrel{\text{def}}{=} 2^{|\widetilde{\mathsf{G}}_i|}$ be the size of the codomain of $\widetilde{\mathsf{G}}_i$, $i \in [m]$. Then for all $i \in [m]$, $r_{<i} \in \operatorname{Supp}(\widetilde{R}_{<i})$ and uniform $\widetilde{R}_i$:*

$$\mathop{\mathrm{E}}_{y_i \leftarrow \widetilde{\mathsf{G}}_i(r_{<i}, \widetilde{R}_i)} \left[\log \frac{1}{\Pr\left[\widehat{Y}_i = y_i | Y_i = y_i, \widehat{R}_{<i} = r_{<i}\right]} \right] \le \log\left(1 + \frac{L_i - 1}{T}\right) .$$

The second claim of the lemma follows after first establishing using Lemma 4.4 and Markov's inequality that:

$$\Pr_{(y_{\le m+1}, r) \leftarrow (\widetilde{Y}_{\le m+1}, \widetilde{R})} \left[\sum_{i=1}^{m} \log\left(\frac{1}{\Pr\left[\widehat{Y}_i = y_i | \widehat{R}_{<i} = r_{<i}, \widehat{Y}_{<i} = y_{<i}\right]} \right) \ge \frac{m \cdot 2^{\ell}}{T\delta' \ln 2} \right] \le \delta'$$

□

Proof of Lemma 4.4. By definition of $\mathsf{Sim}_T^{\widetilde{\mathsf{G}}}$, we have:

$$\Pr\left[\widehat{Y}_i = y_i | Y_i = y_i, \widehat{R}_{<i} = r_{<i}\right] = 1 - \left(1 - \Pr\left[\widetilde{\mathsf{G}}_i(r_{<i}, \widetilde{R}_i) = y_i\right]\right)^T .$$

Applying Jensen's inequality, we have:

$$\begin{aligned} &\mathop{\mathrm{E}}_{y_i \leftarrow \widetilde{\mathsf{G}}_i(r_{<i}, \widetilde{R}_i)} \left[\log\left(\frac{1}{\Pr\left[\widehat{Y}_i = y_i | Y_i = y_i, \widehat{R}_{<i} = r_{<i}\right]} \right) \right] \\ &\le \log \mathop{\mathrm{E}}_{y_i \leftarrow \widetilde{\mathsf{G}}_i(r_{<i}, \widetilde{R}_i)} \left[\frac{1}{\Pr\left[\widehat{Y}_i = y_i | Y_i = y_i, \widehat{R}_{<i} = r_{<i}\right]} \right] \\ &= \log\left(\sum_{y \in \operatorname{Im}(\widetilde{\mathsf{G}}_i(r_{<i}, \cdot))} \frac{p_y}{1 - (1 - p_y)^T} \right) \end{aligned}$$

where $p_y = \Pr\left[\widetilde{\mathsf{G}}_i(r_{<i}, \widetilde{R}_i) = y\right]$. Since the function $x/\left(1 - (1-x)^T\right)$ is convex (see Lemma A.1 in the appendix), the maximum of the expression inside the logarithm over probability distributions $\{p_y\}$ is achieved at the extremal points of the standard probability simplex. Namely, when all but one $p_y \to 0$ and the other one is 1. Since $\lim_{x \to 0} x/1 - (1 - x)^T = 1/T$:

$$\log\left(\sum_{y \in \operatorname{Im}(\widetilde{\mathsf{G}}_i)} \frac{p_y}{1 - (1 - p_y)^T} \right) \le \log\left(1 + (L_i - 1) \cdot \frac{1}{T}\right) .$$

□

Remark. For fixed distribution and generators, in the limit where T grows to infinity, the error term caused by the failure of rejection sampling in time T vanishes. In this case, KL-hardness implies block-KL-hardness without any loss in the hardness parameters.

4.2 Blockwise hardness and inaccessible entropy

We first recall the definition from [HRVW16], slightly adapted to our notations.

Definition 4.5 (Inaccessible Entropy)**.** *Let $(Y_1, \ldots, Y_{m+1})$ be a joint distribution.*[4] *We say that $(Y_1, \ldots, Y_{m+1})$ has t-*inaccessible entropy *Δ if for all $(m+1)$-block online generators $\widetilde{\mathsf{G}}$ running in time t and consistent with $(Y_1, \ldots, Y_{m+1})$:*

$$\sum_{i=1}^{m+1} \left(\mathrm{H}(Y_i|Y_{<i}) - \mathrm{H}(\widetilde{Y}_i|\widetilde{R}_{<i}) \right) > \Delta \ .$$

*where $(\widetilde{Y}_1, \ldots, \widetilde{Y}_{m+1}) = \widetilde{\mathsf{G}}(\widetilde{R}_1, \ldots, \widetilde{R}_{m+1})$ for a uniform $\widetilde{R}_{\le m+1}$. We say that $(Y_1, \ldots, Y_{m+1})$ has (t, δ)-*max-inaccessible entropy *Δ if for all $(m+1)$-block online generators $\widetilde{\mathsf{G}}$ running in time t and consistent with $(Y_1, \ldots, Y_{m+1})$:*

$$\Pr_{\substack{r_{\le m+1} \leftarrow \widetilde{R}_{\le m+1} \\ y_{\le m+1} \leftarrow \widetilde{\mathsf{G}}(r_{\le m+1})}} \left[\sum_{i=1}^{m+1} \left(\mathrm{H}^*_{y_i, y_{<i}}(Y_i|Y_{<i}) - \mathrm{H}^*_{y_i, r_{<i}}\left(\widetilde{Y}_i|\widetilde{R}_{<i}\right) \right) \le \Delta \right] < \delta \ .$$

Unfortunately, one unsatisfactory aspect of Definition 4.5 is that inaccessible entropy can be negative since the generator $\widetilde{\mathsf{G}}$ could have more entropy than $(Y_1, \ldots, Y_{m+1})$: if all the Y_i are independent biased random bits, then a generator $\widetilde{\mathsf{G}}$ outputting unbiased random bits will have negative inaccessible entropy. On the other hand, next-block KL-hardness (Definition 4.1) does not suffer from this drawback.

Moreover, in the specific case where $(Y_1, \ldots, Y_{m+1})$ is a flat distribution[5], then no distribution with the same support can have higher entropy and in this case Definitions 4.1 and 4.5 coincide as stated in the following theorem.

Theorem 4.6. *Let $(Y_1, , \ldots, Y_{m+1})$ be a flat distribution and $\widetilde{\mathsf{G}}$ be an $(m+1)$-block generator consistent with $Y_{\le m+1}$. Then for $\widetilde{Y}_{\le m+1} = \widetilde{\mathsf{G}}(\widetilde{R}_{\le m+1})$ for uniform $\widetilde{R}_{\le m+1}$:*

1. *For every $y_{\le m+1}, r_{\le m+1} \in \mathrm{Supp}(\widetilde{Y}_{\le m+1}, \widetilde{R}_{\le m+1})$, it holds that*

$$\sum_{i=1}^{m+1} \left(\mathrm{H}^*_{y_i, y_{<i}}(Y_i|Y_{<i}) - \mathrm{H}^*_{y_i, r_{<i}}\left(\widetilde{Y}_i|\widetilde{R}_{<i}\right) \right) = \sum_{i=1}^{m+1} \mathrm{KL}^*_{y_i, y_{<i}, r_{<i}}\left(\widetilde{Y}_i|\widetilde{R}_{<i}, \widetilde{Y}_{<i} \,\middle\|\, Y_i|R_{<i}, Y_{<i}\right)$$

In particular, $(Y_1, \ldots, Y_{m+1})$ is (t, Δ)-block-$\mathrm{KL}^{\delta}_{\max}$-hard if and only if it has (t, δ)-max-inaccessible entropy at least Δ.

[4]We write $m+1$ the total number of blocks, since in this section we will think of Y_{m+1} (also written as W) as the witness of a KL-hard distributional search problem and $(Y_1, \ldots, Y_m)$ are the blocks of the instance as in the previous section.

[5]For example, the distribution $(Y_{\le m}, Y_{m+1}) = (f(U), U)$ for a function f and uniform input U is always a flat distribution even if f itself is not regular.

2. Furthermore,

$$\sum_{i=1}^{m+1}\left(\mathrm{H}\left(Y_i|Y_{<i}\right)-\mathrm{H}\left(\widetilde{Y}_i|\widetilde{R}_{<i}\right)\right)=\sum_{i=1}^{m+1}\mathrm{KL}\left(\widetilde{Y}_i|\widetilde{R}_{<i},\widetilde{Y}_{<i}\,\middle\|\,Y_i|R_{<i},Y_{<i}\right),$$

so in particular, $(Y_1,\ldots,Y_{m+1})$ *is* (t,Δ)*-block-*KL*-hard if and only if it has* t*-inaccessible entropy at least* Δ*.*

Proof. For the sample notions, the chain rule (Proposition 2.6) gives:

$$\sum_{i=1}^{m+1}\mathrm{H}^*_{y_i,y_{<i}}\left(Y_i|Y_{<i}\right)=\mathrm{H}^*_{y}\left(Y_{\le m+1}\right)=\log|\operatorname{Supp}(Y_{\le m+1})|$$

for all y since Y is flat. Hence:

$$\begin{aligned}\log|\operatorname{Supp}(Y_{\le m+1})|-\sum_{i=1}^{m+1}\mathrm{H}^*_{y_i,y_{<i}}\left(\widetilde{Y}_i|\widetilde{R}_{<i}\right)&\\ =\sum_{i=1}^{m+1}\left(\mathrm{H}^*_{y_i,y_{<i}}\left(Y_i|Y_{<i}\right)-\mathrm{H}^*_{y_i,r_{<i}}\left(\widetilde{Y}_i|\widetilde{R}_{<i}\right)\right)&\\ =\sum_{i=1}^{m+1}\mathrm{KL}^*_{y_i,y_{<i},r_{<i}}\left(\widetilde{Y}_i|\widetilde{R}_{<i},\widetilde{Y}_{<i}\,\middle\|\,Y_i|R_{<i},Y_{<i}\right)&\ ,\end{aligned}$$

so the second claim follows by taking the expectation over $(\widetilde{Y}_{\le m+1},\widetilde{R}_{\le m+1})$ on both sides. □

By chaining the reductions between the different notions of hardness considered in this work (witness-KL-hardness, block-KL-hardness and inaccessible entropy), we obtain a more modular proof of the theorem of Haitner *et al.* [HRVW16], obtaining inaccessible entropy from any one-way function.

Theorem 4.7. *Let* n *be a security parameter,* $f:\{0,1\}^n\to\{0,1\}^n$ *be a* (t,ε)*-one-way function, and* X *be uniform over* $\{0,1\}^n$*. For* $\ell\in\{1,\ldots,n\}$*, decompose* $f(X)\stackrel{\text{def}}{=}(Y_1,\ldots,Y_{n/\ell})$ *into blocks of length* ℓ*. Then:*

1. *For every* $0\le\Delta\le\log(1/\varepsilon)$*,* $(Y_1,\ldots,Y_{n/\ell},X)$ *has* t'*-inaccessible entropy at least* $(\log(1/\varepsilon)-\Delta)$ *for* $t'=\Omega\left(t\cdot\Delta\cdot\ell^2/(n^2\cdot 2^\ell)\right)$*.*
2. *For every* $0<\delta\le 1$ *and* $0\le\Delta\le\log(1/\varepsilon)-\log(2/\delta)$*,* $(Y_1,\ldots,Y_{n/\ell},X)$ *has* (t',δ)*-max-inaccessible entropy at least* $(\log(1/\varepsilon)-\log(2/\delta)-\Delta)$ *for* $t'=\Omega\left(t\cdot\delta\cdot\Delta\cdot\ell^2/(n^2\cdot 2^\ell)\right)$*.*

Proof. Since f is (t,ε)-one-way, the distributional search problem $\left(\Pi^f,f(X)\right)$ where $\Pi^f=\{(f(x),x):x\in\{0,1\}^n\}$ is (t,ε)-hard. Clearly, $(f(X),X)$ is supported on Π^f, so by applying Theorem 3.8, we have that $(\Pi^f,f(X),X)$ is $(\Omega(t),\log(1/\varepsilon))$-witness-KL-hard and $(\Omega(t),\log(1/\varepsilon)-\log(2/\delta))$-witness-$\mathrm{KL}^{\delta/2}_{\max}$-hard. Thus, by Theorem 4.2 we have that $(Y_1,\ldots,Y_{n/\ell},X)$ is $\left(\Omega\left(t\cdot\Delta\cdot\ell^2/(n^2\cdot 2^\ell)\right),\log(1/\varepsilon)-\Delta\right)$-block-KL-hard and $\left(\Omega\left(t\cdot\delta\cdot\Delta\cdot\ell^2/(n^2\cdot 2^\ell)\right),\log(1/\varepsilon)-\log(2/\delta)-\Delta\right)$-block-$\mathrm{KL}^{\delta}_{\max}$-hard, and we conclude by Theorem 4.6. □

Remark. For comparison, the original proof of [HRVW16] shows that for every $0 < \delta \leq 1$, $(Y_1, \ldots, Y_{n/\ell}, X)$ has (t', δ)-max-inaccessible entropy at least $(\log(1/\varepsilon) - 2\log(1/\delta) - O(1))$ for $t' = \tilde{\Omega}\left(t \cdot \delta \cdot \ell^2/(n^2 \cdot 2^\ell)\right)$, which in particular for fixed t' has quadratically worse dependence on δ in terms of the achieved inaccessible entropy: $\log(1/\varepsilon) - 2 \cdot \log(1/\delta) - O(1)$ rather than our $\log(1/\varepsilon) - 1 \cdot \log(1/\delta) - O(1)$.

Corollary 4.8 (Theorem 4.2 in [HRVW16])**.** *Let n be a security parameter, $f : \{0,1\}^n \to \{0,1\}^n$ be a strong one-way function, and X be uniform over $\{0,1\}^n$. Then for every $\ell = O(\log n)$, $(f(X)_{1\ldots\ell}, \ldots, f(X)_{n-\ell+1\ldots n}, X)$ has $n^{\omega(1)}$-inaccessible entropy $\omega(\log n)$ and $(n^{\omega(1)}, \mathrm{negl}(n))$-max-inaccessible entropy $\omega(\log n)$.*

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A Missing Proofs

Lemma A.1. *For all* $t \geq 1$, $f : x \mapsto \frac{x}{1-(1-x)^t}$ *is convex over* $[0,1]$.

Proof. We instead show convexity of $\tilde{f} : x \mapsto f(1-x)$. A straightforward computation gives:

$$\tilde{f}''(x) = \frac{x^{t-2}t\big(t(1-x)(x^t+1) - (1+x)(1-x^t)\big)}{(1-x^t)^3}$$

so that it suffices to show the non-negativity of $g(x) = t(1-x)(x^t+1) - (1+x)(1-x^t)$ over $[0,1]$. The function g has second derivative $t(1-x)(t^2-1)x^{t-2}$, which is non-negative when $x \in [0,1]$, and thus the first derivative g' is non-decreasing. Also, the first derivative at 1 is equal to zero, so that g' is non-positive over $[0,1]$ and hence g is non-increasing over this interval. Since $g(1) = 0$, this implies that g is non-negative over $[0,1]$ and f is convex as desired. □

Theorem A.2 (Theorem 3.8 restated)**.** *Let* Π *be a binary relation and let* (Y, W) *be pair of random variables supported on* Π*. If* (Π, Y) *is* (t, ε)*-hard, then* (Π, Y, W) *is* (t', Δ')*-witness-*KL*-hard and* (t', Δ'')*-witness-*$\mathrm{KL}^{\delta}_{\max}$*-hard for every* $\delta \in [0,1]$ *where* $t' = \Omega(t)$*,* $\Delta' = \log(1/\varepsilon)$ *and* $\Delta'' = \log(\delta/\varepsilon)$*.*

Proof. We proceed similarly to the proof of Theorem 3.3. Let $(\widetilde{\mathsf{G}}, \mathsf{S})$ be a pair of algorithms with $\widetilde{\mathsf{G}} = (\widetilde{\mathsf{G}}_1, \widetilde{\mathsf{G}}_{\mathsf{w}})$ a two-block generator supported on Π. Define the linear-time oracle algorithm $\mathsf{A}^{\widetilde{\mathsf{G}}_{\mathsf{w}},\mathsf{S}}(y) \stackrel{\text{def}}{=} \widetilde{\mathsf{G}}_{\mathsf{w}}(\mathsf{S}(y))$. Then

$$\begin{aligned}
\Pr\left[\Pi\big(Y,\mathsf{A}^{\widetilde{\mathsf{G}}_{\mathsf{w}},\mathsf{S}}(Y)\big)\right] &= \Pr\left[\Pi(Y,\widetilde{\mathsf{G}}_{\mathsf{w}}(\mathsf{S}(Y)))\right] \\
&\geq \Pr\left[\widetilde{\mathsf{G}}_1(\mathsf{S}(Y)) = Y\right] && (\widetilde{\mathsf{G}} \text{ is supported on } \Pi) \\
&= \sum_{r \in \operatorname{Supp}(\widetilde{R})} \Pr\left[\mathsf{S}(Y) = r \wedge Y = \widetilde{\mathsf{G}}_1(r)\right] \\
&\geq \sum_{\substack{r \in \operatorname{Supp}(\widetilde{R}) \\ w \in \operatorname{Supp}(\widetilde{\mathsf{G}}_2(\widetilde{R}))}} \Pr\left[\mathsf{S}(Y) = r \wedge Y = \widetilde{\mathsf{G}}_1(r) \wedge W = w\right] \\
&= \mathop{\mathrm{E}}_{\substack{r \leftarrow \widetilde{R} \\ w \leftarrow \widetilde{\mathsf{G}}_2(r)}} \left[\frac{\Pr\left[\mathsf{S}(Y) = r \wedge Y = \widetilde{\mathsf{G}}_1(r) \wedge W = w\right]}{\Pr\left[\widetilde{R} = r \wedge \widetilde{\mathsf{G}}_2(r) = w\right]}\right] \\
&= \mathop{\mathrm{E}}_{\substack{r \leftarrow \widetilde{R} \\ y \leftarrow \widetilde{\mathsf{G}}_1(r) \\ w \leftarrow \widetilde{\mathsf{G}}_2(r)}} \left[2^{-\operatorname{KL}^*_{r,y,w}\left(\widetilde{R},\widetilde{\mathsf{G}}_1(\widetilde{R}),\widetilde{\mathsf{G}}_2(\widetilde{R}) \,\middle\|\, \mathsf{S}(Y),Y,W\right)}\right],
\end{aligned}$$

The witness-KL-hardness then follows by applying Jensen's inequality (since 2^{-x} is convex) and the witness-$\mathrm{KL}_{\mathrm{max}}$-hardness follows by Markov's inequality by considering the event that the sample-KL is smaller than Δ (this event has density at least δ). □