Announcements

- start recording
- scribe: work w/other scribe to produce one set of notes
- My OH: Mon 12:30-1:30, Thu 9-10
- TF section/OH: Mon 2-3, Tue 8-9, Wed 5-6, Thu 5:30-6:30
- Fixing Problem 3.3 now extra credit
- If Zoom goes down, check Piazza
- Sync whiteboard
- Use spreadsheet (linked from Piazza) to find project partners.
- Jamboard link in chat

Agenda

- Explicit Constructions of Expanders
- Undirected S-T Connectivity in Logspace

Recall: randomness-efficient error reduction using expanders

Given a randomized algorithm A that uses n random bits (e.g. power method)

- choose $x^{(1)}, \dots, x^{(t)} \in \{0,1\}^n$

random walk on an expander $G=(V,E)$

$N = |V| = 2^n \qquad V \Leftrightarrow \{0,1\}^n$

- choose
$x^{(1)} \xleftarrow{R} V$
$x^{(2)} \xleftarrow{R} \{x^{(1)}\text{'s } d \text{ neighbors}\}$
$x^{(3)} \xleftarrow{R} \{x^{(2)}\text{'s } d \text{ neighbors}\}$
$\vdots$
$x^{(t)} \xleftarrow{R} \{x^{(t-1)}\text{'s } d \text{ neighbors}\}$

random bits $= n + O(t \log d)$

$$= O(n)$$

$$\Uparrow$$

$t = O(n)$, $d = O(1)$

$= O(\log N)$

<u>Thm</u>: If G has spectral expansion $\gamma = 1 - \omega$ and $V_1, \dots, V_t$ are a random walk on G w/ uniform start vertex V_1, then

$$\forall B \quad \Pr_{r.w.}\left[\bigwedge_{j=1}^{t} (V_j \in B) \right] \le \left(\mu + \omega \cdot (1-\mu) \right)^t$$

where $\mu = \dfrac{|B|}{|V|}$

There is also a Chernoff Bound for Expander Walks $\rightarrow$ reduce 2-sided error via majority vote / median

Def (spectral norm): $\|M\| = \max_{x \neq 0} \frac{\|Mx\|}{\|x\|}$ = largest singular value of M

Matrix Decomposition

Lemma: G has spectral expansion γ iff

$$W = \gamma J + (1-\gamma) E$$

where J = all $1/n$ matrix

and E = some matrix w/ $\|E\| \leq 1$.

i.e. $Wv = \gamma v^{\parallel} + (1-\gamma) e$ where $v^{\parallel}$ = projection of v in direction of $\vec{u}$

and $\|e\| \leq \|v\|$

cf. Vector Decomposition Technique

$$v = v^{\parallel} + v^{\perp}$$

$$Wv = (Wv)^{\parallel} + (Wv)^{\perp}$$

$$= \underbrace{v^{\parallel}} + e \quad , \text{ where } \|e\| \leq (1-\gamma)\|v^{\perp}\|$$

and $e \perp \vec{u}$

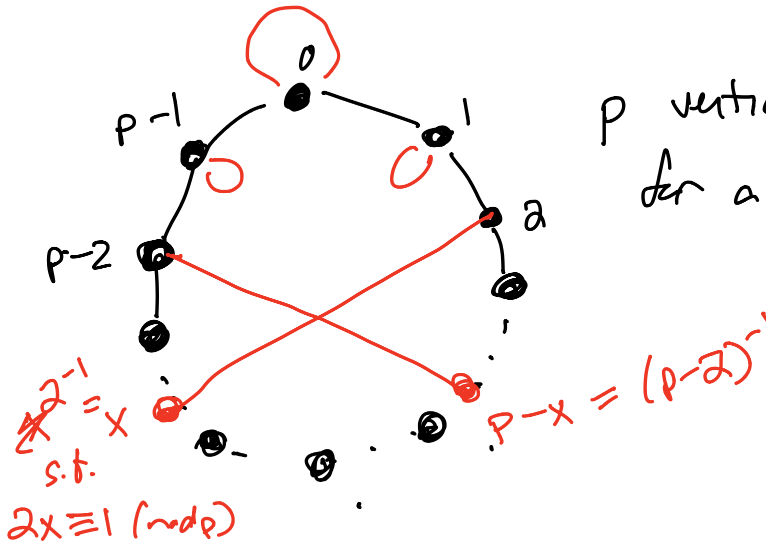
Explicit Constructions of Expanders

Goal: infinite family of graphs $\mathcal{G} = \{G_i\}$ s.t.

- $\exists$ constant d s.t. every G_i is d-regular
- $\exists$ constant $\gamma > 0$ s.t. every G_i has spectral expansion at least γ
- given $a \in \{1, \ldots, n_i\}$ and $j \in \{1, \ldots, d\}$ can compute j^{th} neighbor of vertex a in G_i in time $\text{poly}(\log n_i)$ "fully explicit"
- the family $\{n_i\}$ of sizes is not too sparse ($\rightarrow$ can convert into a family of expanders of all sizes)

$n_i = \#$ vertices in G_i

Example:



p vertices for a prime p

- Proof of expansion: deep number theory
- ≈ "Schreier graph" of a Non-abelian group
  $$\mathrm{Sch}\left(SL_2(\mathbb{Z}_p),\ P^1(\mathbb{Z}_p),\ \left\{\begin{pmatrix}1 & \pm 1\\ 0 & 1\end{pmatrix},\ \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right\}\right)$$
- Abelian groups can't give constant-degree expanders (PS3)

Our Approach

- start w/a "constant-sized expander" eg from PS3 problem 4
- repeatedly apply graph ops to get larger expanders

(n, d, γ)-graph: n vertices, degree d, spectral expansion $\geq \gamma$

Squaring: $(n, d, 1-\omega) \longmapsto (n, d^2, 1-\omega^2)$

$(n, d, \gamma) \longmapsto (n, d^2, 2\gamma - \gamma^2)$

Tensoring: $(n, d, \gamma) \longmapsto (n^2, d^2, \gamma)$

$G_1 \otimes G_2$: → vertex set $V = V_1 \times V_2$

→ edge weights $w_{(a_1,a_2),(b_1,b_2)} = w_{(a_1,b_1)} \cdot w_{(a_2,b_2)}$

→ adjacency matrix $M = M_1 \otimes M_2$

→ random-walk matrix $W = W_1 \otimes W_2$

→ eigenvectors of $W = \left(v_1 \otimes v_2 \;:\; \begin{array}{l} v_1 \text{ e-vector of } W_1 \\ v_2 \text{ e-vector of } W_2 \end{array} \right)$

$(v \otimes w)(a,b) = v(a)\, w(b)$

→ eigenvalues/singular values of $W = \left(\alpha_1 \alpha_2 \;:\; \begin{array}{l} \alpha_1 \text{ e-value of } W_1 \\ \alpha_2 \text{ e-value of } W_2 \end{array} \right)$

→ spectral expansion $= \min\{\gamma(G_1), \gamma(G_2)\}$

$1 \cdot 1 \quad \omega(G_1) \cdot 1$

$1 \cdot \omega(G_2)$

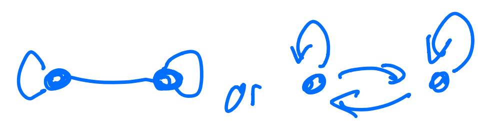
Exercise for Breakouts

Let G_1 be a regular digraph w/ adjacency matrix M_1

Draw the graph G_2 in each case below.

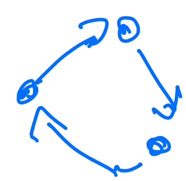
$M = M_1 \otimes M_2$ | G_2

$$M_1 \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} M_1 & M_1 \\ M_1 & M_1 \end{pmatrix}$$

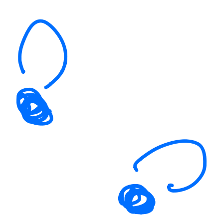


$$M_1 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & M_1 \\ M_1 & 0 \end{pmatrix}$$

$$M_1 \otimes \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & M_1 \\ M_1 & 0 & 0 \\ 0 & M_1 & 0 \end{pmatrix}$$



$$\begin{pmatrix} M_1(1,1) & 0 & M_1(1,2) & 0 & \cdots & M_1(1,n) & 0 \\ 0 & M_1(1,1) & 0 & M_1(1,2) & & 0 & M_1(1,n) \\ \vdots & & \ddots & & & & \vdots \\ 0 & M_1(n,1) & & \cdots & & M_1(n,n) & 0 \\ M_1(n,1) & 0 & & & & 0 & M_1(n,n) \end{pmatrix}$$



$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\cong \qquad = (I_2 \otimes M_1)$

$$\begin{pmatrix} M_1 & 0 \\ 0 & M_1 \end{pmatrix} = M_1 \otimes I_2$$

Zig-zag Product: $(n_1, d_1, \gamma_1) \circledz (d_1, d_2, \gamma_2) \mapsto (n_1 d_1, d_2^2, \gamma_1 \gamma_2^2)$

$\gamma_1 = 1-\omega_1 \quad \gamma_2 = 1-\omega_2 \Rightarrow \gamma' = 1-\omega$

for $\omega \leq \omega_1 + 2\omega_2$

Proof: $W_{zigzag} = \tilde{W}_2 \hat{W}_1 \tilde{W}_2$

$\tilde{W}_2 = I_{n_1} \otimes W_2 = I_{n_1} \otimes (\gamma_2 J_{n_2} + (1-\gamma_2) E_2)$

$\hat{W}_1$ = permutation matrix

$$\hat{W}_1((b,j),(a,i)) = \begin{cases} 1 & \text{if } i^{th} \text{ edge leaving } a = j^{th} \text{ edge entering } b \\ 0 & \end{cases}$$

$$W_{zigzag} = \underbrace{\gamma_2^2 \tilde{J}_{n_2} \hat{W}_1 \tilde{J}_{n_2}} + \underline{(1-\gamma_2^2) \cdot F}$$

$$\tilde{W}_2 = \gamma_2 (I_{n_1} \otimes J_{n_2}) + (1-\gamma_2) \cdot (I_{n_2} \otimes E_2)$$

$$\tilde{J}_{n_2} \hat{W}_1 \tilde{J}_{n_2} = W_1 \otimes J_{n_2}$$

γ_1

Constructing Expanders

Let $H = (d^4, d, 7/8)$ graph (PS 3)

$G_1 = H^2$

$G_{t+1} = G_t^2 ⓩ H$ $\qquad n_t =$

$\omega_{t+1} \leq$

$\text{Time}(G_{t+1}) = 2\,\text{Time}(G_t) + O(\log n_t)$

to compute neighbors

S-T Connectivity

Given $G=(V,E)$, $s,t \in V$, is there a path from s to t?

- Directed G: · time and space $O(n)$ (DFS)
  - · or space $O(\log^2 n)$ (M^n via repeated squaring)

- Undirected G: · in randomized space $O(\log n)$ (PS2, 1979)

Here: deterministic space $O(\log n)$ [Reingold 2004]

$G_0 =$ d^4-regular, aperiodic modification of G

$G_{t+1} = G_t^2 ⓩ H$ $\quad$ H a $(d^4, d, 3/4)$-graph

$\gamma_0 \geq$

$\gamma_{t+1} \geq$

$\gamma_{O(\log n)} \geq \Omega(1) \implies G_{O(\log n)}$ a constant-degree expander.

$\text{Space}(G_{t+1}) = \text{Space}(G_t) + O(1)$.

Digraphs w/ poly(n) mixing time: Complete for RL