Announcements

- start recording
- scribe: work w/other scribe to produce one set of notes
- My OH: ~~Mon 12:30-1:30~~ Tue 4:30-5:30, Thu 9-10 (Mon. holiday)
- TF hybrid section/OH: Mon 2-3, Wed 2-3, Wed 5-6, Thu 5:30-6:30
- PS 2 due Fri, including final project ideas problem
- If Zoom goes down, check Piazza
- Sync whiteboard
- Among Us Fri night
- No jamboard today
- Last chance to fill out feedback – Q on Perusall + readings was supposed to be "how useful"

Agenda

- Power Method (finish)
- Expander Graphs: Measures of Expansion

Fiedler's Algorithm for Sparsest Cut

Given weighted, undirected G, find $S \subseteq V$ w/ $d(S) \leq d(V)/2$

and $\phi(S) \lesssim \sqrt{2\nu_2} = O(\sqrt{\phi(G)})$

1. Find a vector $y \perp \vec{d}$

   s.t. $\dfrac{y^T L y}{y^T D y} \lesssim \nu_2$ $\Big\}$ by eigendecomposition of N

   time $O(n^\omega)$

   $2 \leq \omega \leq 2.378\ldots$

2. Sort $y(1) \leq y(2) \leq \cdots \leq y(n)$ $\}$ $O(n \log n)$

3. Find $k \in \{1, \ldots, n\}$ minimizing

   $$\frac{w(e(\{1, \ldots, k\}, \{k+1, \ldots, n\}))}{\min\{d(\{1, \ldots, k\}), d(\{k+1, \ldots, n\})\}}$$ $\Big\}$ $O(m)$

   where $e(S,T) = \{(a,b) : a \in S, b \in T\}$

How to speed up Step 1?

Goal: time $\tilde{O}(n+m)$

Power Method

I. Estimate Largest E-Value + corresponding test vector of a PSD matrix M

$m = \#$ nonzero entries in M

① Choose $x \xleftarrow{R} \{\pm 1\}^n$

② let $y = M^k x$ for $k = O\left(\frac{\log(n/\varepsilon)}{\varepsilon}\right)$

time $O(k \cdot m)$

Thm: with constant probability, $\mu_1 \geq \frac{y^T M y}{y^T y} \geq (1-\varepsilon) \cdot \mu_1$

II. 2nd largest eigenvalue of a psd matrix

① Let $x \stackrel{R}{\leftarrow} \{\pm 1\}^n$

② $x_0 = x - \langle x, v_1 \rangle v_1$

③ $y = M^k x_0$ for $k = O\left(\frac{\log(n/\varepsilon)}{\varepsilon}\right)$

Cor: w/ constant probability

$$\mu_2 \geq \frac{y^T M y}{y^T y} \geq \mu_2 \cdot (1-\varepsilon)$$

and $y \perp v_1$

III. Second Smallest E-value of Normalized Laplacian

Apply II to matrix

$$2I - N = I + A = 2D^{-1/2}\tilde{W}D^{1/2} \quad \text{psd}$$

$\uparrow$ normalized laplacian $\quad$ $\uparrow$ normalized adj mx

E-values: $2,\ 1+\omega_2,\ 1+\omega_3, \ldots,\ 1+\omega_n$

$\qquad\qquad\quad \| \qquad\quad \|$

$\qquad\quad 2-\nu_2 \quad 2-\nu_3 \quad \ldots$

Obtain vector $y \perp \overrightarrow{d^{1/2}}$ s.t.

$$2 - \frac{y^T N y}{y^T y} = \frac{y^T(2I-N)y}{y^T y} \geq (1-\varepsilon)(2-\nu_2)$$

$$\text{i.e.} \quad \frac{y^T N y}{y^T y} \leq (1+\varepsilon)\underline{\nu_2} + 2\varepsilon$$

Take $\varepsilon = \frac{\nu_2}{4}$

$$\Rightarrow \text{time } O\left(m \cdot \frac{\log(n/\nu_2)}{\nu_2}\right)$$

Better: apply power meth to "pseudoinverse" N^+

eigenvalues $0, \frac{1}{\nu_2}, \frac{1}{\nu_3}, \ldots, \frac{1}{\nu_n}$

$(1-\varepsilon)(2-\nu_2)$

$(1-\varepsilon)\frac{1}{\nu_2} \approx \frac{1}{(1+\varepsilon)\nu_2}$

Later in course: algorithms that given G and and x, compute N^+x in time $\tilde{O}(m+n)$

$\Rightarrow$ find cut of conductance $O(\sqrt{\nu_2})$ in time $\underbrace{\tilde{O}(m+n)}_{\text{nearly linear time}}$

Expander Graphs

Many applications in TCS!

d-regular n-vertex unweighted digraphs, $n \to \infty$

SPARSE: typically $d = O(1)$

(sometimes $d = \mathrm{polylog}(n)$)

WELL-CONNECTED: several defs

1) spectral expansion γ : $\gamma(G) = 1 - \omega(G) \geq \gamma$ $\in [0,1]$

$$\omega(G) \stackrel{\text{def}}{=} \omega_u(G) = \max_p \frac{\|W_p - u\|}{\|p - u\|} = \max_{x \perp u} \frac{\|Wx\|}{\|x\|} = \max\{\omega_2, -\omega_n\} \geq \omega_2$$

u = uniform distribution $= \vec{1}/n$ $\quad \gamma \leq \gamma_2$

~~directed n-cycle~~
~~$\omega(G) = 1$~~
~~$\gamma(G) = 0$~~

Typically want $\gamma > 0$ independent of n.

~~undirected n-cycle~~
~~$\gamma(G) = \Theta(\frac{1}{n^2})$~~

Ideally maximize $\gamma = 1 - \omega$ as a function of d

complete graph
$\gamma(G) = 1 - \frac{1}{n-1}$ ✓ (but not sparse)

2) <u>(k, a) vertex expansion</u>

$\forall S \ |S| \le k, \quad |N(S)| \ge a \cdot |S|$

neigh

Typically $k = \Omega(n)$, $a = 1 + \Omega(1)$

Ideally <u>maximize a</u> as fnctn of n, k, d.

$a \ge 1$ by regularity

e.g. $a = d - 1.01$

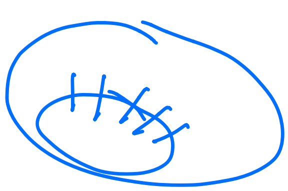
3) <u>(k, ε) edge expansion</u>

$\forall S \ |S| \le k, \quad |e(S, S^c)| \ge \varepsilon \cdot d \cdot |S|$

Typically $k = \Omega(n)$, $\varepsilon = \Omega(1)$

Ideally maximize ε as func. of n, k, d

$$\phi(S) = \frac{|e(S, S^c)|}{d \cdot |S|}$$



Edge Expansion vs. Spectral Expansion

Thm: Let $\mathcal{G}$ be an infinite family of regular undirected, lazy graphs,

The following are equivalent

1) $\exists\ \gamma > 0$ s.t. every $G \in \mathcal{G}$ has spectral expansion at least γ.

$\Downarrow$ w/o laziness $\qquad \Uparrow$ laziness necessary

2) $\exists\ \varepsilon > 0$ s.t. every $G \in \mathcal{G}$ is an $(N/2, \varepsilon)$ edge expander.

consider bipartite $K_{N/2, N/2}$

Proof: By Cheeger's Inequalities

if G lazy

$$\frac{\gamma(G)}{2} \underset{\uparrow}{\leq} \frac{\nu_2}{2} \leq \phi(G) \leq \sqrt{2\nu_2} \leq \sqrt{2\gamma(G)}$$

$$\phi(G) = \min\{\varepsilon : G \text{ is } (N/2, \varepsilon) \text{ edge expander}\}$$

$$\gamma(G) \overset{\text{def}}{=} 1 - \omega(G) = 1 - \max\{\omega_2, -\omega_n\}$$

$$= \min\{1 - \omega_2,\ 1 + \omega_n\}$$

$$= \min\{\nu_2,\ 1 + \omega_n\}$$

$$\overset{\text{lazy}}{=} \nu_2 \qquad (\omega_n \geq 0 \text{ if } G \text{ lazy})$$

Vertex Expansion vs. Spectral Expansion

d a constant

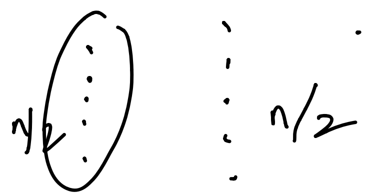
Thm: Let $\mathcal{G}$ be an infinite family of d-regular undirected graphs.

The following are equivalent — "expander graphs"

1) $\exists\ \gamma > 0$ s.t. every $G \in \mathcal{G}$ has spectral expansion at least γ.

$\Downarrow$ next time $\qquad \Uparrow$ PS3?

2) $\exists\ \varepsilon > 0$ s.t. every $G \in \mathcal{G}$ is an $(N/2, 1+\varepsilon)$ vertex expander.



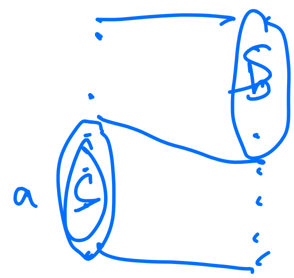
bipartite graphs are not vertex expanders

$$\gamma(G) \overset{\text{def}}{=} 1 - \omega(G) = 1 - \max\{\omega_2, -\omega_n\}$$

$$= \min\{1 - \omega_2,\ 1 + \omega_n\}$$

$$= \min\{\nu_2,\ 1 + \omega_n\}$$

$$\overset{\text{lazy}}{=} \nu_2 \qquad (\omega_n \ge 0 \text{ if } G \text{ lazy})$$



$a+b \leq n/2$

$$e(S, S^c)$$

$$= a \cdot \left(\frac{n}{2} - b\right) + b \cdot \left(\frac{n}{2} - a\right)$$

$$= (a+b) \cdot \frac{n}{2} - \underline{2ab}$$