Announcements

- start recording
- scribe: work w/other scribe to produce one set of notes
- My OH: Mon 12:30-1:30, Thu 9-10
- TF hybrid section/OH: Mon 2-3, Wed 2-3, Wed 5-6, Thu 5:30-6:30
- PS 2 due Fri, including final project ideas problem
- If Zoom goes down, check Piazza
- sync whiteboard
- gather after class: link in chat
- jamboard link in chat
- FILL OUT FEEDBACK FORM

Agenda

- Recap
- Fiedler's Algorithm
- The Power Method

Recap: $\phi(S) = \dfrac{w(\partial S)}{d(S)}$

$$\phi(G) = \min_{S : d(S) \le d(V)/2} \phi(S) = \min_{S \subseteq V} \max \{\phi(S), \phi(V-S)\}$$

$$\phi_k(G) = \min_{\substack{S_1, \dots, S_k \\ \text{disjoint}}} \max \{\phi(S_1), \dots, \phi(S_k)\}$$

Cheeger's Inequalities $\qquad \nu_2/2 \le \phi(G) \le \sqrt{2\nu_2}$

Higher-order Cheeger $\qquad \nu_k/2 \le \phi_k(G) \le \mathrm{poly}(k) \cdot \sqrt{\nu_k}$

Fiedler's Algorithm for Sparsest Cut

Given weighted, undirected G, find $S \subseteq V$ w/ $d(S) \le d(V)/2$

and $\phi(S) \lesssim \sqrt{2\nu_2} = O(\sqrt{\phi(G)})$

① Find a vector $y \perp \vec{d}$

s.t. $\dfrac{y^T L y}{y^T D y} \lesssim \nu_2$ $\Big\}$ by eigendecomposition of N

time $\boxed{O(n^{\omega})}$
$\boxed{2 \le \omega \le 2.378\dots}$

② Sort $y(1) \le y(2) \le \dots \le y(n)$ $\}$ $O(n \log n)$

③ Find $k \in \{1, \dots, n\}$ minimizing $\quad \}$ $O(\cancel{(n)} \cdot m) \to d(1)$ time

$$\frac{w(e(\{1,\dots,k\},\{k+1,\dots,n\}))}{\min\{d(\{1,\dots,k\}),d(\{k+1,\dots,n\})\}}$$

where $e(S,T)=\{(a,b): a\in S, b\in T\}$

$\int w(e(\{\{1\},\{2,\dots,n-1\}\}))$

$\downarrow d(2)$ time

$w/ e(\{1,2\},\{3,\dots,n-1\}))$

How to speed up Step 1?

Goal: time $\tilde{O}(n+m)$

Power Method

I. Largest (Estimate) E-Value + corresponding test vector of a PSD matrix M

$m = \#$ nonzero entries in M

① Choose $x \leftarrow^R \{\pm 1\}^n$ — proxy for choosing x from unit sphere — $\tilde{O}(n)$

② let $y = M^k x$ for $k = O\left(\frac{\log(n/\varepsilon)}{\varepsilon}\right)$ — time $O(k\cdot m)$

cf. repeated squaring $O((\log k)\cdot n^{\omega})$

Thm: with constant probability (e.g. $\geq 3/16$; boost to .99 by repetition), $\mu_1 \geq \frac{y^T M y}{y^T y} \geq (1-\varepsilon)\cdot\mu_1$

Pf: Write $x = c_1 v_1 + \dots + c_n v_n$

$v_1,\dots,v_n$ orthonormal eigenbasis for M

w/ e-values $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n$

$y = M^k x = c_1\mu_1^k v_1 + \dots + c_n \mu_n^k v_n$

Claim: w/ constant prob., $c_1 \geq 1/2$

Then: $y^T y = c_1^2 \mu_1^{2k} + \dots + c_n^2 \mu_n^{2k}$

$\leq \sum_{i=1}^{l} c_i^2 \mu_i^{2k} + \sum_{i=l+1}^{n} c_i^2 \left((1-\varepsilon)\cdot\mu_1\right)^{2k}$

Let l be s.t. $\mu_l \geq (1-\varepsilon)\cdot\mu_1$, $\mu_{l+1} < (1-\varepsilon)\cdot\mu_1$

$\|x\|^2 = n$

$= \sum_i c_i^2$

$$\le \sum_{i=1}^{\ell} c_i^2 \mu_i^{2k} + n \cdot \mu_1^{2k} \cdot \left(\frac{\varepsilon}{n}\right) \qquad (1-\varepsilon)^{2k} \le \frac{\varepsilon}{n}$$

$$\le \sum_{i=1}^{\ell} c_i^2 \mu_i^{2k} + 4\varepsilon c_1^2 \mu_1^{2k}$$

when $c_1 \ge 1/2$

$$\le (1+4\varepsilon)\cdot\left(c_1^2 \mu_1^{2k} + \cdots + c_\ell^2 \mu_\ell^{2k}\right)$$

$$y^T M y = c_1^2 \mu_1^{2k+1} + \cdots + c_n^2 \mu_n^{2n+1}$$

$$\ge c_1^2 \mu_1^{2k+1} + \cdots + c_\ell^2 \mu_\ell^{2k+1}$$

$$\ge (1-\varepsilon)\cdot \mu_1 \cdot \left(c_1^2 \mu_1^{2k} + \cdots + c_\ell^2 \cdot \mu_\ell^{2k}\right)$$

$$\text{Ratio} \ge \frac{1-\varepsilon}{1+4\varepsilon} = 1 - O(\varepsilon)$$

sketch
Pf of claim:

$$c_1 = x^T v_1 = \sum_a x(a) v_1(a)$$

$x(a)$'s are random + independent ± 1 r.v.s

$$\mathop{E}_x[c_1] = \sum_a 0 \cdot v_1(a) = 0$$

We want: anti-concentration

$E[x(a)^3] = 0$

$$\mathop{E}_x[c_1^2] = \sum_{a,b} E[x(a)\, x(b)\, v_1(a)\, v_1(b)]$$

zero when $a \ne b$

$$= \sum_a E[x(a)^2\, v_1(a)^2] = \|v_1\|^2 = 1$$

$x(a)^2 \to 1$

$$\mathop{E}_x[c_1^4] \le 3$$

$\mathrm{Var}[c_1^2] \le 2$

Q: why not use Hoeffding or Chebychev?

$\mathrm{Stdev}(c_1^2) = \sqrt{2}$

instead: Paley-Zygmund

Breakout Q: Why doesn't the above theorem/proof show that the lazy random walk on the undirected n-cycle mixes in time $O(\log n)$?

Reasons it might apply

- $\widetilde{W}$ on regular graph is PSD
- $\langle x, v_1\rangle \geq \frac{\|x\|}{\mathrm{poly}(n)}$ also holds for any prob. dist $\vec{x}$ and $v_1 = \vec{1}/\sqrt{n}$

Reasons it doesn't

$\frac{y^T M y}{y^T y} \geq (1-\varepsilon)\mu_1$ does not imply that y is close to v_1

e.g. $y = v_2$ and $\mu_2 \geq (1-\varepsilon)\mu_1$

II. 2nd largest eigenvalue of a psd matrix

$\mu_1 = 1$

$\mu_2 = 1 - \Theta\left(\frac{1}{n^2}\right)$

① Let $x \leftarrow \{\pm 1\}^n$

②

③ $y = M^k x_0$ for $k = O\left(\frac{\log(n/\varepsilon)}{\varepsilon}\right)$

Cor: w/ constant probability

$$\mu_2 \geq \frac{y^T M y}{y^T y} \geq \mu_2 \cdot (1-\varepsilon)$$

and $y \perp v_1$

III. Second Smallest $\mathcal{E}$-value of Normalized Laplacian

Apply II to matrix

$\mathcal{E}$-values :

Obtain vector $y \perp$ s.t.

$$\frac{y^T \qquad y}{y^T y} \geq$$

i.e. $\dfrac{y^T N y}{y^T y} \leq$

Take $\varepsilon = \dfrac{\nu_2}{4}$

$\Rightarrow$ time $O\left(m \cdot \dfrac{\log(n/\nu_2)}{\nu_2}\right)$

Not good

Better: apply power meth to "pseudoinverse" N^+

eigenvalues $0, \frac{1}{\nu_2}, \frac{1}{\nu_3}, \dots, \frac{1}{\nu_n}$

Later in course: algorithms that given G and and x, compute $N^+ x$ in time $\tilde{O}(m+n)$

$\Rightarrow$ find cut of conductance $O(\sqrt{\nu_2})$ in time $\underbrace{\tilde{O}(m+n)}_{\text{nearly linear time}}$