

Problem Set 0

Harvard SEAS - Fall 2020

Due: Fri. Sep. 11, 2020 (5pm)

This problem set is to help you review background material and gauge your preparation for the course. You are not expected to have seen all of these problems before, only to have the background needed for their solutions. It is optional but highly recommended. Your solutions will be graded for the sake of feedback, but the grades will not count.

Your problem set solutions must be typed (in e.g. L^AT_EX) and submitted on Gradescope. You are allowed 12 late days for the semester, of which at most 5 can be used on any individual problem set. (1 late day = 24 hours exactly). Please name your file `ps0-lastname.*`.

In general, the problem sets in the course may require a lot of thought, so be sure to start them early. You are encouraged to discuss the course material and the homework problems with each other in small groups (2-3 people). Identify your collaborators on your submission. Discussion of homework problems may include brainstorming and verbally walking through possible solutions, but should not include one person telling the others how to solve the problem. In addition, each person must write up their solutions independently, and these write-ups should not be checked against each other or passed around.

Strive for clarity and conciseness in your solutions, emphasizing the main ideas over low-level details. Do not despair if you cannot solve all the problems! Difficult problems are included to stimulate your thinking and for your enjoyment, not to overwork you. *'ed problems are extra credit.

Problem 1.(Similar Matrices) An $n \times n$ matrix A is called *similar* to an $n \times n$ matrix B if there exists a non-singular matrix Q such that $B = Q^{-1}AQ$. Show that if A and B are similar, then they share the same eigenvalues.

Problem 2.(Graph Theory)

- (a) Recall that a *proper k -coloring* of a graph G is an assignment of k colors to the vertices of G such that every two neighboring vertices have different colors. Finding a proper coloring using the minimal number of colors is NP-hard, but we can efficiently color a graph with few colors when the graph has low degree. Specifically, give a linear-time algorithm that given a graph G , finds a proper $(\Delta + 1)$ -coloring of G , where Δ is the maximum degree of any vertex in G . (Here linear time means time $O(m)$, where m is the number of edges in G .)
- (b) Recall that an *independent set* in a graph G is a set of vertices with no edge between them. Show that every graph G of n vertices and maximum degree Δ has an independent set of size at least $n/(\Delta + 1)$.

Problem 3.(Positive Definite Matrices) An $n \times n$ real symmetric matrix M is called *positive definite (pd)* if for every non-zero vector $x \in \mathbb{R}^n$, $x^T M x > 0$. Prove the following facts about pd symmetric matrices. You may use the Spectral Theorem for real symmetric matrices.

- (a) A symmetric matrix M is pd iff all of its eigenvalues are positive.

- (b) Every pd symmetric matrix M is invertible.
- (c) For every (possibly asymmetric) matrix M , M is invertible iff the symmetric matrix $M^T M$ is pd.
- (d) Let M be an $n \times n$ pd symmetric matrix and consider any partition of M into blocks

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where A is $m \times m$, B is $m \times (n - m)$, and C is $(n - m) \times m$, and D is $(n - m) \times (n - m)$. Then the block A and the *Schur Complement* $S = D - CA^{-1}B$ are both pd symmetric matrices.

Problem 4.(Matrix Inversion Reduces to Matrix Multiplication)

- (a) Let M be a block matrix of the form

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

such that both A and $S = D - CA^{-1}B$ are invertible. Prove that M is invertible with

$$M^{-1} = \begin{bmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{bmatrix}.$$

- (b) By directly applying the definition, two $n \times n$ matrices can be multiplied using $O(n^3)$ arithmetic operations. In CS124, you saw that $n \times n$ matrices can be multiplied using $O(n^\omega)$ arithmetic operations for some $\omega < 3$. Specifically, Strassen's algorithm shows that $\omega \leq \log_2 7 \approx 2.81$. The best known bound on ω is $\omega \leq 2.3729$ (by Virginia Vassilevska Williams in 2012 and François Le Gall in 2014). Using the formula in Part (a) and Problem 3, derive an algorithm that inverts pd symmetric matrices using $O(n^\omega)$ arithmetic operations.
- (c) Combine Part (b) and Problem 3(c) to derive an algorithm to invert arbitrary invertible matrices using $O(n^\omega)$ arithmetic operations.

Note: counting arithmetic operations does not tell the full story about computational complexity. We should also be concerned about the bitlengths of the entries (e.g. when inverting integer or rational matrices exactly) and/or how error accumulates (e.g. when using floating-point approximations to real numbers), and additional tricks are needed to control these.